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BEHAVIOUR OF log $\eta(\tau)$ AT A RATIONAL POINT

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Dedekind[3], using his transformation formula for log $\eta(\tau)$, and Hardy[4], using contour integration, determined the limiting behaviour of log $\eta(r + iy)$ as $y \to 0+$ where r is, a fixed rational number. In this paper, we give an elementary proof of this result with an improved error estimate. Our method is a refinement of Hardy's technique and may be applied to other classes of Lambert series for which no such transformation theory exists. We also show that Dedekind's transformation formula yields a sharper form of our main result.

1. INTRODUCTION

Riemann, in his Fragmente XXVIII published posthumously, determined the limiting behaviour of the elliptic modular functions (cf. Magnus *et al.*[5], Chapter X) $\log k$, $\log k'$, $\log \frac{2K}{\pi}$ as the parameter q approaches a rational place on the unit circle.

Let a, b be coprime positive integers, Im $\tau > 0$, $\eta(\tau)$, the Dedekind η-function, $\log \eta(\tau) = \pi i \tau/12 - \sum_{n=1}^{\infty} q^{2n}/n(1 - q^{2n})$, $q = \exp(\pi i \tau)$ and $f(y) = -\pi/12a^2 y - \frac{1}{2} \log(ay) - i\left(s\left(b, a\right) - \frac{b}{12a}\right)$ where $s(b, a)$ is the classical Dedekind sum defined by

$$s(b, a) = \sum_{\mu=1}^{a} \left(\left(\frac{\mu}{a}\right)\right)\left(\left(\frac{b\mu}{a}\right)\right).$$

Here $((x)) = 0$ if x is an integer and $x - [x] - \frac{1}{2}$ otherwise, $[x]$ being the largest integer $\leqslant x$. Further, let $\log \eta(b/a + iy) = f(y) + g(y)$ for $y > 0$.

In order to justify Riemann's results, Dedekind[3], using his transformation formula for log $\eta(\tau)$, and Hardy[4], using a sophisticated technique in contour integration, proved that as $y \to 0+$

$$g(y) = o(1). \qquad \ldots(1.1)$$

The object of this paper is to obtain a more precise form of (1.1).

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Theorem—For each positive integer n, we have, as $y \to 0+$

$$g(y) = O(y^n).$$

Our method of proof is a refinement of Hardy's technique and is completely elementary. We dispense with contour integration and use instead Euler-Maclaurin summation formula. We believe that our method may be applied to other classes of Lambert series for which no such transformation formulae exist.

In section 4, we deduce, from Dedekind's transformation formula for $\log \eta(\tau)$, that as $y \to 0+$

$$g(y) = O(e^{-2\pi/a^2 y}) \qquad \ldots(1.2)$$

where the constant 2π is best possible.

2. Further Notation and Lemmas

We write, as in Hardy[4]

$$q = e^{\pi i \tau} = e^{-\pi y} e^{\pi i b/a} = \rho e^{\pi i b/a}$$

$$\sigma = \rho^{2a},\ e^{-\alpha} = \sigma \text{ (so that } \alpha = 2\pi a y) \text{ and}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{2n}}{1-q^{2n}} = \sum_{t=1}^{\infty} \frac{1}{ta} \frac{\rho^{2ta}}{1-\rho^{2ta}}$$

$$+ \sum_{s=1}^{a-1} \sum_{t=0}^{\infty} \frac{1}{ta+s} \frac{\rho^{2(ta+s)} e^{2sb\pi i/a}}{1-\rho^{2(ta+s)} e^{2sb\pi i/a}}$$

$$= T + \sum_{s=1}^{a-1} T_s. \qquad \ldots(2.1)$$

Also let

$$T = \frac{1}{a} \sum_{t=1}^{\infty} \frac{\sigma^t}{t(1-\sigma^t)}$$

$$= \frac{2}{a} \sum_{t=1}^{\infty} \frac{\sigma^t}{t(1-\sigma^{2t})} - \frac{1}{a} \sum_{t=1}^{\infty} \frac{\sigma^t}{t(1+\sigma^t)}$$

$$= \frac{1}{a} (2T_1 - T_2). \qquad \ldots(2.2)$$

Finally for $1 \leqslant s \leqslant a-1$, let $\lambda = \lambda(s) = \dfrac{2sb\pi}{a}$, $s = aw$ and

$$T_s = \frac{1}{a}\sum_{t=0}^{\infty} \frac{1}{t+w} \quad \frac{e^{-\alpha(t+w)+i\lambda}}{1-e^{-\alpha(t+w)+i\lambda}}$$

$$= \frac{1}{a}\left(T_s' - T_s'' \right) \qquad \text{...(2.3)}$$

where

$$T_s' = \frac{e^{i\lambda}}{1-e^{i\lambda}} \sum_{t=0}^{\infty} \frac{e^{-\alpha(t+w)}}{t+w} \qquad \text{...(2.4)}$$

and

$$T_s'' = \frac{e^{2i\lambda}}{1-e^{i\lambda}} \sum_{t=0}^{\infty} \frac{e^{-\alpha(t+w)}\,(1-e^{-\alpha(t+w)})}{(t+w)\,(1-e^{-\alpha(t+w)+i\lambda})} . \qquad \text{...(2.5)}$$

The following lemma is basic to our work and is of independent interest.

Lemma 2.1—Let n be a positive integer, $f : [0, \infty) \to \mathbb{C}$ be $(2n + 1)$ times continuously differentiable, $f^{(k)}(x) \to 0$ as $x \to \infty$ for $0 \leqslant k \leqslant 2n - 1$ and $\int_0^\infty | f^{(2n+1)}(x) | \, dx$ be finite. Then for fixed $w \geqslant 0$ and $r > 0$, we have

$$\sum_{t=0}^{\infty} f(r(t+w)) = \frac{1}{r}\int_0^\infty f(x)\,dx - \sum_{k=1}^{2n} \frac{B_k(w)\, f^{(k-1)}(0)}{k!} r^{k-1} + O(r^{2n})$$

where $B_k(x)$ denotes the Bernoulli polynomial of order k defined by

$$\frac{ze^{xz}}{e^z-1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k, \; | z | < 2\pi.$$

PROOF : The well-known Euler-Maclaurin Summation formula (cf. de Bruij[2], Chapter 3) states that if n and N are positive integers $g : [0, \infty) \to \mathbb{C}$ is $(2n + 1)$ times continuously differentiable, then

$$\sum_{t=0}^{N} g(t) = \int_0^N g(x)\,dx + \tfrac{1}{2}(g(0) + g(N))$$

$$+ \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!}\left(g^{(2k-1)}(N) - g^{(2k-1)}(0)\right) + \int_0^N P_{2n+1}(t)\, g^{(2n+1)}(t)\,dt$$

where $B_k = B_k(0)$ and $P_k(x)$ is the periodic Bernoulli function of order k.

Hence for fixed $r > 0$, we have by hypothesis

$$\sum_{t=0}^{\infty} f(r(t+w)) = \int_0^{\infty} f(r(x+w))\,dx + \tfrac{1}{2} f(rw)$$

$$- \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(rw)\, r^{2k-1} + O(r^{2n}). \qquad \dots(2.6)$$

However, by Taylor's formula

$$\int_0^{\infty} f(r(x+w))\,dx = 1/r \int_{rw}^{\infty} f(y)\,dy$$

$$= \frac{1}{r}\int_0^{\infty} f(x)\,dx - \frac{1}{r}\int_0^{rw} f(x)\,dx$$

$$= \frac{1}{r}\int_0^{\infty} f(x)\,dx - \frac{1}{r}\int_0^{rw} \{f(0) + f^{(1)}(0)\,x + \dots + \frac{f^{(2n-1)}(0)}{(2n-1)!}$$

$$\times x^{2n-1} + \frac{f^{(2n)}(y_x)}{(2n)!} x^{2n}\}\,dx,\ y_x \in (0, rw)$$

$$= \frac{1}{r}\int_0^{\infty} f(x)\,dx - \sum_{k=0}^{2n-1} \frac{f^{(k)}(0)}{(k+1)!} w^{k+1} r^k + O(r^{2n}) \qquad \dots(2.7)$$

$$\tfrac{1}{2} f(rw) = \tfrac{1}{2} \sum_{k=0}^{2n-1} \frac{f^{(k)}(0)}{k!} w^k r^k + O(r^{2n}) \qquad \dots(2.8)$$

and

$$- \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(rw)\, r^{2k-1}$$

$$= - \sum_{k=1}^{n} \frac{B_{2k}}{(2k)}\Big[\sum_{i=2k-1}^{2n} \frac{f^{(i)}(0)}{(i-2k+1)!} (rw)^{i-2k+1} + O(r^{2n-2k+2})\Big] r^{2k-1}$$

$$= - \sum_{i=1}^{2n-1} \Big[\sum_{k=1}^{\min\left(\frac{i+1}{2},\, n\right)} \binom{i+1}{2k} B_{2k}\, w^{i+1-2k} \Big] \frac{f^{(i)}(0)}{(i+1)!} r^i + O(r^{2n})$$

(equation continued on p. 115)

$$= - \sum_{i=1}^{2n-1} \left[B_{i+1}(w) - w^{i+1} + \frac{i+1}{2} w^i \right] \frac{f^{(i)}(0)\, r^i}{(i+1)!} + O(r^{2n}) \quad \ldots(2.9)$$

where in the last step we used the fact that $B_k(x) = \sum_{i=0}^{n} \binom{k}{i} B_i x^{k-i}$ (cf. Magnus *et al.*[5], p. 25). Since $B_1(x) = x - \frac{1}{2}$, the lemma follows from (2.6) through (2.9) on simplification.

In the following n denotes a fixed integer $\geqslant 1$, the symbol $O(\ )$ refers to the passage as $y \to 0+$ and the order constant depends at most on parameters other than y. We recall that $\alpha = 2\pi a y$.

Lemma 2.2—$\frac{2}{a} T_1 = \frac{\pi^2}{6a\alpha} - \frac{\log 2}{a} + \frac{\alpha}{12a} + O(\alpha^{2n+1})$.

PROOF : It is easy to see that the function $f : [0, \infty) \to \mathbb{C}$ defined by $f(x) = \frac{1}{x}\left(\frac{1}{\sinh x} - \frac{1}{x} \right)$ for $x > 0$ and $f(0) = -\frac{1}{6}$ satisfies the conditions of Lemma 2.1. Since $f(x)$ is an even function, we have $f^{(2k-1)}(0) = 0$ for all $k \geqslant 1$ and further (cf. Hardy[4], p. 83)

$$\int_0^\infty \frac{1}{x}\left(\frac{1}{\sinh x} - \frac{1}{x} \right) dx = -\log 2. \quad \ldots(2.10)$$

Hence by Lemma 2.1 with $w = 0$, we obtain

$$\sum_{t=0}^{\infty} f(\alpha t) = -\frac{\log 2}{\alpha} - \frac{1}{12} + O(\alpha^{2n}).$$

Now the lemma follows on noting that :

$$\frac{2}{a} T_1 = \frac{1}{a} \sum_{t=1}^{\infty} \frac{1}{t \sinh(\alpha t)}$$

$$= \frac{\alpha}{a} \left(\sum_{t=0}^{\infty} f(\alpha t) - f(0) \right) + \frac{\pi^2}{6a\alpha}.$$

Lemma 2.3—$\frac{1}{a} T_2 = \frac{1}{2a} \log\left(\frac{\pi}{2} \right) - \frac{\log \alpha}{2a} + \frac{\alpha}{8a} + O(\alpha^{2n+1})$.

PROOF : We observe that the function $f(x) = \frac{e^{-x}(1 - e^{-x})}{x(1 + e^{-x})}$ for $x > 0$ and

$f(0) = \frac{1}{2}$ satisfies the conditions of Lemma 2.1. Also we have

$f(x) = \dfrac{1-e^{-x}}{x} - \dfrac{1-e^{-x}}{x(1+e^{-x})} = \dfrac{1-e^{-x}}{x}$ an even function, so that $f^{(2k-1)}(0) = -\dfrac{1}{2k}$ for all $k \geqslant 1$ and further (cf. Hardy[4], p. 84)

$$\int_0^\infty \frac{e^{-x}(1-e^{-x})}{x(1+e^{-x})}\, dx = \log\left(\frac{\pi}{2}\right).$$

Finally since (cf. Magnus *et al.*[5], p. 36)

$$\log\left(\frac{1-e^{-x}}{2}\right) = \sum_{k=1}^{2n-1} \frac{B_k}{k\,(k!)} x^k + O(x^{2n}),\ 0 < x < 1 \qquad \ldots(2.11)$$

the result follows from Lemma 2.1 with $w = 0$ on noting that

$$\frac{1}{a} T_2 = \frac{1}{a} \sum_{t=1}^{\infty} \frac{\sigma^t}{2t}\left(\frac{1-\sigma^t}{1+\sigma^t} + 1\right)$$

$$= \frac{1}{2a} \sum_{t=1}^{\infty} \frac{e^{-\alpha t}(1-e^{-\alpha t})}{t(1+e^{-\alpha t})} - \frac{1}{2a} \log(1-e^{-\alpha})$$

$$= \frac{\alpha}{2a} \sum_{t=0}^{\infty} f(\alpha t) - \frac{\alpha}{4a} - \frac{1}{2a} \log(1-e^{-\alpha}).$$

Lemma 2.4— $T = \dfrac{\pi^2}{6a\alpha} + \dfrac{\log(\alpha/2\pi)}{2a} - \dfrac{\alpha}{24a} + O(\alpha^{2n+1}).$

PROOF : Follows from (2.2) and Lemmas 2.2 and 2.3.

Lemma 2.5— $-\dfrac{1}{a} T''_{s} = -\left(\dfrac{a-1}{2a}\right)\log(2\pi) + \frac{1}{2}\log a$

$$+ \frac{1}{2} \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k\,(k!)}\left(\frac{1}{a^k} - \frac{1}{a}\right) B_k \alpha^k$$

$$+ \frac{i}{2a} \sum_{k=1}^{2n} \frac{(-1)^k}{k\,(k!)}\left[\sum_{s=1}^{a-1} \cot\left(\frac{\pi b s}{a}\right) B_k\left(\frac{s}{a}\right)\right] \alpha^k + O(\alpha^{2n+1}).$$

PROOF : We note that the function $f_s(x) = \dfrac{e^{-x}(1-e^{-x})}{x\,(1-e^{-x+i\lambda})}$ for $x \neq 0$ and

$f(0) = \dfrac{1}{1 - e^{i\lambda}}$ satisfies the conditions of Lemma 2.1 and hence

$$-\frac{1}{a}\sum_{s=1}^{a-1} T_s'' = -\frac{1}{a}\sum_{s=1}^{a-1} \frac{e^{2i\lambda}}{1-e^{i\lambda}}\int_0^\infty \frac{e^{-x}(1-e^{-x})}{x(1-e^{-x+i\lambda})}\,dx$$

$$+\frac{1}{a}\sum_{k=1}^{2n}\frac{\alpha^k}{k!}\; g_k^{(k-1)}(0) + O(\alpha^{2n+1}) \quad \text{.. (2.12)}$$

where

$$g_k(x) = \sum_{s=1}^{a-1} B_k\left(\frac{s}{a}\right)\frac{e^{2i\lambda}}{1-e^{i\lambda}} f_s(x).$$

Now it is known (cf. Hardy[4], pp 80-81) that

$$-\frac{1}{a}\sum_{s=1}^{a-1}\frac{e^{2i\lambda}}{1-e^{i\lambda}}\int_0^\infty f_s(x)\,dx = -\left(\frac{a-1}{2a}\right)\log(2\pi) + \tfrac{1}{2}\log a. \quad \text{...(2.13)}$$

To evaluate $g_k^{(k-1)}(0)$, we write

$$g_k(x) = -\left[\sum_{s=1}^{a-1}\frac{e^{i\lambda}}{1-e^{i\lambda}}B_k\left(\frac{s}{a}\right)\right]\left(\frac{1-e^{-x}}{x}\right) + h_k(x)$$

where

$$h_k(x) = \sum_{s=1}^{a-1}\frac{e^{i\lambda}}{1-e^{i\lambda}}\;B_k\left(\frac{s}{a}\right)\left(\frac{1-e^{-x}}{x(1-e^{-x+i\lambda})}\right).$$

Since $B_n(1-x) = (-1)^n\,B_n(x)$ (cf. Magnus *et al.*[5], p. 25), we have $h_k(-x) = (-1)^k\,h_k(x)$ and consequently $h_k^{(k-1)}(0) = 0$ for all k. Thus

$$g_k^{(k-1)}(0) = \left[\sum_{s=1}^{a-1}\frac{e^{i\lambda}}{1-e^{i\lambda}}\,B_k\left(\frac{s}{a}\right)\right]\left(\frac{(-1)^k}{k}\right)$$

$$= \frac{(-1)^k}{k}\sum_{s=1}^{a-1}\left[-\tfrac{1}{2}+\tfrac{1}{2}\cot\left(\frac{\pi b s}{a}\right)\right]B_k\left(\frac{s}{a}\right)$$

(*equation continued on p.* 118)

$$= \frac{(-1)^{k+1}}{2k} \left(\frac{1}{a^{k-1}} - 1 \right) B_k + \frac{i(-1)^k}{2k} \sum_{s=1}^{a-1} \cot\left(\frac{\pi bs}{a} \right) B_k \left(\frac{s}{a} \right) \tag{2.14}$$

in view of $\sum_{k=1}^{m-1} B_p \left(\frac{k}{m} \right) = (m^{1-p} - 1) B_p$ (cf Magnus *et al.*[5], p. 26). Now the lemma follows from (2.12), (2.13) and (2.14).

Lemma 2.6— $\operatorname{Re}\left(\frac{1}{a} \sum_{s=1}^{a-1} T'_s \right) = \left(\frac{a-1}{2a} \right) \log \alpha - \frac{1}{2} \log a$

$$+ \frac{1}{2} \sum_{k=1}^{2n} \frac{B_k}{k\,(k!)} \left(\frac{1}{a^k} - \frac{1}{a} \right) \alpha^k$$

$$+ O(\alpha^{2n+1}).$$

PROOF : We have

$$\operatorname{Re}\left(\frac{1}{a} \sum_{s=1}^{a-1} T'_s \right) = -\frac{1}{2} \sum_{s=1}^{a-1} \sum_{t=0}^{\infty} \frac{\rho^2\,(at+s)}{at+s}$$

$$= -\frac{1}{2} \sum_{\substack{n=1 \\ n \not\equiv 0 \ (\mathrm{mod}\ a)}}^{\infty} \frac{\rho^{2n}}{n} = \frac{1}{2} \{\log(1-\rho^2) - \log(1-\rho^{2a})\}.$$

Hence the lemma follows from (2.11).

Lemma 2.7— $\operatorname{Im}\left(\frac{1}{a} \sum_{s=1}^{a-1} T'_s \right) = \pi s\,(b, a)$

$$+ \frac{1}{2a} \sum_{k=1}^{2n} \frac{1}{k\,(k!)} \left(\sum_{s=1}^{a-1} \cot\left(\frac{\pi bs}{a} \right) B_k\left(\frac{s}{a} \right) \right)$$

$$\times \alpha^k + O(\alpha^{2n+1}).$$

PROOF : First we recall that[3]

$$\int_0^1 \frac{\sum_{s=1}^{a-1} \cot\left(\frac{\pi bs}{a} \right) x^{2s-1}}{1 - x^{2a}}\, dx = \pi s\,(b, a).$$

Now since $\rho = e^{-\pi y} = e^{-\alpha/2a}$ and $w = s/a$, we have

$$\operatorname{Im}\left(\frac{1}{a}\sum_{s=1}^{a-1} T'_s\right) = \frac{1}{2a}\sum_{s=1}^{a-1}\cot\left(\frac{\pi bs}{a}\right)\sum_{t=0}^{\infty}\frac{e^{-\alpha\left(t+\frac{s}{a}\right)}}{\left(t+\frac{s}{a}\right)}$$

$$= \sum_{s=1}^{a-1}\cot\left(\frac{\pi bs}{a}\right)\sum_{t=0}^{\infty}\int_0^{\rho} x^{2(at+s)-1}\,dx$$

$$= \int_0^{\rho}\frac{\sum_{s=1}^{a-1}\cot\left(\frac{\pi bs}{a}\right)x^{2s-1}}{1-x^{2a}}\,dx$$

$$= \pi s(b, a) - \int_{\rho}^{1}\frac{\sum_{s=1}^{a-1}\cot\left(\frac{\pi bs}{a}\right)x^{2s-1}}{1-x^{2a}}\,dx$$

$$= \pi s(b, a) - \frac{1}{2a}\int_0^{\alpha}\frac{\sum_{s=1}^{a-1}\cot\left(\frac{\pi bs}{a}\right)e^{\left(1-\frac{s}{a}\right)v}}{e^{v}-1}\,dv$$

$$= \pi s(b,a) + \frac{1}{2a}\int_0^{\alpha}\frac{\sum_{s=1}^{a-1}\cot\left(\frac{\pi bs}{a}\right)e^{\left(\frac{s}{a}\right)v}}{e^{v}-1}\,dv$$

$$= \pi s(b, a) + \frac{1}{2a}\int_0^{\alpha}\sum_{s=1}^{a-1}\cot\left(\frac{\pi bs}{a}\right)\left(\sum_{k=0}^{2n}\frac{B_k\left(\frac{s}{a}\right)}{k!}\right.$$

$$\left. v^{k-1} + O(v^{2n})\right)dv$$

$$= \pi s(b, a) + \frac{1}{2a}\int_0^{\alpha}\sum_{s=1}^{a-1}\cot\left(\frac{\pi bs}{a}\right)$$

$$\left(\sum_{k=1}^{2n}\frac{B_k\left(\frac{s}{a}\right)}{k!}v^{k-1} + O(v^{2n})\right)dv$$

and the lemma follows.

3. Proof of the Theorem

Since $\alpha = 2\pi a y$, we have from (2.1) through (2.5) and Lemmas 2.2 through 2.7

$$\sum_{k=1}^{\infty} \frac{1}{k} \frac{q^{2k}}{1-q^{2k}} = \frac{\pi}{12a^2y} + \tfrac{1}{2}\log(ay) - \frac{\pi}{12}y + \pi i s(b,a) + O(y^{2n+1}).$$

Here $q = e^{-\pi y + \pi i b/a}$, a, b are coprime positive integers, $y > 0$ and n is a fixed positive integer. Now the theorem follows.

Proof of (1.2)

Dedekind's transformation formula for $\log \eta(\tau)$ (cf. Apostol[1], Chapter 3) states that if a, b, c, d are integers satisfying $ad - bc = 1$ and $c > 0$, then for $\mathrm{Im}\ \tau > 0$

$$\log \eta\left(\frac{a\tau+b}{c\tau+d}\right) = \log \eta(\tau) + \tfrac{1}{2}\log\{-i(c\tau+d)\} + \pi i\left(\frac{a+d}{12c} + s(-d,c)\right). \quad \ldots (4.1)$$

Now let h, k and H be positive integers, $(h,k) = 1$, $hH \equiv -1 \pmod{k}$ and $y > 0$. On taking $a = H$, $c = k$, $d = -h$, $b = -\dfrac{hH+1}{k}$ and $\tau = \dfrac{iy+h}{k}$ in (4.1), we obtain, in view of $\dfrac{a\tau+b}{c\tau+d} = \dfrac{H}{k} + \dfrac{i}{ky}$,

$$\log \eta\left(\frac{h}{k} + i\frac{y}{k}\right) = -\tfrac{1}{2}\log y + \log \eta\left(\frac{H}{k} + \frac{i}{ky}\right) - \pi i\left(\frac{H-h}{12k} + s(h,k)\right).$$

On writing ky for y, this gives

$$\log \eta\left(\frac{h}{k} + iy\right) = -\frac{\pi}{12k^2y} \tfrac{1}{2}\log(ky) - \pi i\left(s(h,k) - \frac{h}{12k}\right) - \sum_{n=1}^{\infty} \frac{\exp\left\{2\pi i n\left(\frac{H}{k} + \frac{i}{k^2y}\right)\right\}}{n\left(1 - \exp\left\{2\pi\, i n\left(\frac{H}{k} + \frac{i}{k^2y}\right)\right\}\right)}.$$

Now since $\exp\left\{2\pi\, in\left(\frac{H}{k} + \frac{i}{k^2y}\right)\right\} / n\left(1 - \exp\left(2\pi i n\left(\frac{H}{k} + \frac{i}{k^2y}\right)\right)\right)$ $= O(\exp(-2\pi n/k^2y)/(1 - \exp(-2\pi/k^2y)) = O(\exp(-2\pi n/k^2y))$ as $y \to 0$, (1.2) follows.

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ON CONVEX SEMI-METRIC SPACES AND GENERALISED SHANNON INEQUALITIES

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Symmetric and non-symmetric convex semi-metric spaces are defined and a number of examples of each are given. In this process, some generalised Shannon inequalities and some Shannon-type inequalities are obtained.

1. INTRODUCTION

We consider a set X of ordered n-tuples $x = (x_1, x_2, \ldots, x_n)$ with each $x_i > 0$ and a 'discrepancy' function $d(x, y)$ defined on X with the following possible properties:

(i) $d(x, y) \geqslant 0$

(ii) $d(x, y) = 0$ iff $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) + d(y, z) \geqslant d(x, z)$

(v) $d(x, y)$ is a convex function of both x and y.

The pair (X, d) is called a 'metric space' (MS) if (i)—(iv) are satisfied. It will be called a 'convex metric space' (CMS) if (i) — (v) are satisfied, a 'convex semi-metric space' (CSMS) if (i) — (iii) and (v) are satisfied and a 'convex non-symmetric semi-metric space' (CNSMS) if (i), (ii) and (v) alone are satisfied.

The triangle inequality (iv) is important for some mathematical purposes, but the condition (v) is important for applications to maximum entropy and minimum-discrimination-information models in many fields of science and technology[5,10].

We shall consider some examples of discrepancy functions $d(x, y)$ satisfying (i), (ii) and (v). It may be noted that if $d(x, y)$ is such a function, then

$$D(x, y) = d(x, y) + d(y, x) \qquad \ldots(1)$$

will satisfy (i), (ii), (iii) and (v).

In order to show that in our examples (i) is satisfied, we need certain inequalities of the Shannon type [1,8,10]. Some of these are generalised Shannon's inequalities in the sense that these include Shannon's inequality as a special or as a limiting case. We

discuss some of these generalised Shannon inequalities and also give some Shannon-type inequalities.

2. A General Convex Non-Symmetric Semi-Metric Space

Consider the special case when $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1$, so that x and y represent probability distributions and let

$$d(x, y) = \sum_{i=1}^{n} y_i \varphi\left(\frac{x_i}{y_i}\right), \qquad \ldots(2)$$

where $\varphi(\,.\,)$ is a twice-differentiable convex function for which $\varphi(1) = 0$. This function (2) was first introduced by Csiszer[5] as a measure of directed divergence. By Jensen's inequality[4] for convex functions

$$\sum_{i=1}^{n} y_i \varphi\left(\frac{x_i}{y_i}\right) \geqslant \varphi\left(\sum_{i=1}^{n} y_i \frac{x_i}{y_i}\right) = \varphi\left(\sum_{i=1}^{n} x_i\right) = \varphi(1) = 0 \qquad \ldots(3)$$

so that condition (i) is satisfied. Also when $x_i = y_i$ for all i, $d(x, y) = 0$. To minimize $d(x, y)$ regarded as a function of $x_1, x_2, \ldots, x_n$, subject to

$$x_1 + x_2 + x_3 + \ldots + x_n = y_1 + y_2 + \ldots + y_n = 1 \qquad \ldots(4)$$

we form the Lagrangian

$$L = \sum_{i=1}^{n} y_i \varphi\left(\frac{x_i}{y_i}\right) - \lambda\left(\sum_{i=1}^{n} x_i - 1\right). \qquad \ldots(5)$$

Using Lagrange's method, we get

$$\varphi'\left(\frac{x_1}{y_1}\right) = \varphi'\left(\frac{x_2}{y_2}\right) = \ldots = \varphi'\left(\frac{x_n}{y_n}\right). \qquad \ldots(6)$$

Since $\varphi(x)$ is convex, for each value of $\varphi'(x)$, there is only one value of x, so that (6) gives

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \ldots = \frac{x_n}{y_n} = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} = 1 \qquad \ldots(7)$$

so that an extreme value of $d(x, y)$ occurs when $x_i = y_i$ for all i and since $\varphi(1) = 0$, this extreme value is zero. From (3) it is then easily shown that this is the globally minimum value of $d(x, y)$.

Again since $y_i > 0$, $d(x, y)$ is a convex function of x_i and its local minimum subject to (4) is its global minimum. Also

$$\frac{\partial^2 d}{\partial x_i^2} = \frac{1}{y_i} \varphi'' \left(\frac{x_i}{y_i} \right) > 0, \quad \frac{\partial^2 d}{\partial x_i \partial x_j} = 0 \qquad \ldots(8)$$

$$\frac{\partial^2 d}{\partial y_i^2} = \frac{x_i^2}{y_i^3} \varphi'' \left(\frac{x_i}{y_i} \right) > 0, \quad \frac{\partial^2 d}{\partial y_i \partial y_j} = 0 \qquad \ldots(9)$$

so that the Hessian matrices of the second order partial derivatives of $d(x, y)$ with respect to both $x_1, x_2, \ldots x_n$ and $y_1, y_2, \ldots, y_n$ are positive definite and $d(x, y)$ is a convex function of both x and y.

Thus $d(x, y)$, defined by (2) satisfies condition (i), (ii) and (v) so that (X, d) is a convex non-symmetric semi-metric space. However if

$$D(x, y) = \sum_{i=1}^{n} y_i \varphi \left(\frac{x_i}{y_i} \right) + \sum_{i=1}^{n} x_i \varphi \left(\frac{y_i}{x_i} \right) \qquad \ldots(10)$$

then (X, D) is a convex symmetric semi-metric space.

3. Special Cases

Using $\sum_{i=1}^{n} x_i = 1$, $\sum_{i=1}^{n} y_i = 1$, we get the following special cases of (2) :

(a) $\varphi(x) = x^2 - 1$, $d(x, y) = \sum_{i=1}^{n} (y_i \frac{x_i^2}{y_i^2} - 1) = \sum_{i=1}^{n} \frac{(x_i - y_i)^2}{y_i}$ ...(11)

(b) $\varphi(x) = \frac{1}{x} - 1$, $d(x, y) = \sum_{i=1}^{n} y_i \left(\frac{y_i}{x_i} - 1 \right) = \sum_{i=1}^{n} \frac{(y_i - x_i)^2}{x_i}$ ...(12)

(c) $\varphi(x) = \frac{x^\alpha - x}{\alpha - 1}$, $\alpha > 0, \alpha \neq 1$;

$$d(x, y) = \frac{1}{\alpha - 1} \left[\sum_{i=1}^{n} x_i^\alpha y_i^{1-\alpha} - 1 \right]. \qquad \ldots(13)$$

Our discussion then shows that

$$\frac{1}{\alpha - 1} \left[\sum_{i=1}^{n} x_i^\alpha y_i^{1-\alpha} - 1 \right] \geqslant 0 \text{ when } \alpha > 0; \alpha \neq 1 \qquad \ldots(14)$$

and the equality sign holds iff $x_i = y_i$ for all i.

(d) If we let $\alpha \to 1$ in (13), we get

$$\varphi(x) = x \ln x,\ d(x, y) = \sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} \qquad \text{...(15)}$$

so that

$$\sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} \geqslant 0. \qquad \text{...(16)}$$

Inequality (16) is called Shannon's inequality and, in this sense, (14) may be called a generalized Shannon inequality since (16) is a limiting case of (14). Inequality (14) can also be written as

$$\sum_{i=1}^{n} \frac{x_i^{\alpha}}{y_i^{\alpha-1}} \geqslant 1 \text{ when } \alpha > 1;\ \sum_{i=1}^{n} \frac{x_i^{\alpha}}{y_i^{\alpha-1}} \leqslant 1 \text{ when } 0 < \alpha < 1 \qquad \text{...(17)}$$

provided $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1$.

Inequality (14) may also be called Renyi's inequality or Havrda and Charvat's inequality since the measures of directed divergence given by Renyi[4] and Havrda and Charvat[7], Kapur[11] are

$$\frac{1}{\alpha-1} \ln \sum_{i=1}^{n} x_i^{\alpha}\ y_i^{1-\alpha} \text{ and } \frac{1}{\alpha-1}\left[\sum_{i=1}^{n} x_i^{\alpha}\ y_i^{1-\alpha} - 1\right]$$

$$\alpha > 0,\ a \neq 1 \qquad \text{...(18)}$$

respectively and these have to be greater than or equal to zero. Similarly Shannon's inequality corresponds to the measure of directed divergence of Kullback and Leibler[12].

(e) $\varphi(x) = (\sqrt{x} - 1)^2,\ d(x, y) = \sum_{i=1}^{n} (\sqrt{x_i} - \sqrt{y_i})^2$ . ...(19)

This corresponds to Bhattacharya distance (Behara and Nath[2]).

(f) $\varphi(x) = (x^{1/2j} - 1)^{2j},\ d(x,y) = \sum_{i=1}^{n} \left(x_i^{1/2j} - y_i^{1/2j} \right)^{2j}$; ...(20)

where j is a positive integer

This corresponds to generalized Bhattacharya distance

(g) The most general form of generalized Shannon's inequality obtained so far is

$$\sum_{i=1}^{n} y_i \varphi\left(\frac{x_i}{y_i}\right) \geqslant 0 \qquad \text{...(21)}$$

of which the others are particular cases.

We may point out that while corresponding to any given continuous convex function $\varphi(x)$ with $\varphi(1) = 0$, there is a unique $d(x, y)$, a given $d(x, y)$ can be obtained from an infinity of functions $\varphi(x)$. Thus if $\varphi(x)$ leads to $d(x, y)$, then $\varphi(x) + k(x - 1)$, where k is any arbitrary real number, also leads to the same $d(x, y)$ since

$$\sum_{i=1}^{n} y_i\left(\varphi\left(\frac{x_i}{y_i}\right) + k\left(\frac{x_i}{y_i} - 1\right)\right) = \sum_{i=1}^{n} y_i \varphi\left(\frac{x_i}{y_i}\right)$$

$$+ k \sum_{i=1}^{n} (x_i - y_i) = \sum_{i=1}^{n} y_i \varphi\left(\frac{x_i}{y_i}\right).$$

Thus two continuous convex functions, both vanishing at $x = 1$ and differing by a multiple of $(x - 1)$ lead to the same $d(x, y)$.

In particular $\varphi_1(2)(x) = x^2 - 1$, $\varphi_2(x) = (x - 1)^2$, $\varphi_3(x) = (x^2 - 1) + a(x \quad 1)$, $\varphi_4(x) = (x - 1)^2 + b(x - 1)$ all lead to (11).

However our main interest is not in $\varphi(x)$ but it is in $d(x, y)$ to which it leads and in the inequality which arises due to the non-negativity of $d(x, y)$. $\varphi(x)$ is only a means of arriving at $d(x, y)$; the goal is $d(x, y)$. If many functions $\varphi(x)$ lead to the same $d(x, y)$, it is all right from the point of view of convex semi-metric spaces

4. An Elementary Proof of Renyi's Inequality

The above proof depends on Jensen's inequality for convex functions. One other proof was given by Kapur[11]. Another proof based on Holder's inequality has been given in Aczel and Daroczy[1].

We give below an alternative proof which is elementary in the sense that it does not depend on a prior knowledge of any standard inequality. Let

$$f(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} \; y_i^{1-\alpha} - 1 \qquad \text{...(22)}$$

then

$$f'(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} \; y_i^{1-\alpha} \ln \frac{x_i}{y_i} \qquad \text{...(23)}$$

$$f''(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} \quad y_i^{1-\alpha} \left(\ln \frac{x_i}{y_i}\right)^2 \geqslant 0. \qquad \ldots(24)$$

Thus $f(\alpha)$ is a convex function of α. Let

$$y_i = x_i\,(1 + \epsilon_i), \qquad \ldots(25)$$

so that

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i \Rightarrow \sum_{i=1}^{n} x_i\,\epsilon_i = 0 \qquad \ldots(26)$$

then

$$f'(\alpha) = -\sum_{i=1}^{n} x_i\,(1 + \epsilon_i)^{1-\alpha} \ln(1 + \epsilon_i). \qquad \ldots(27)$$

Using (26),

$$f'(\alpha) = -\sum_{i=1}^{n} x_i\,[(1 + \epsilon_i)^{1-\alpha} \ln(1 + \epsilon_i) - \epsilon_i]\,. \qquad \ldots(28)$$

It is easily shown that coefficient of x_i in (28) is $\geqslant 0$ if $\alpha < 0$ and is $\leqslant 0$ if $\alpha > 1$ so what $f(\alpha)$ is a decreasing function of α when $\alpha < 0$ and is an increasing function of α when $\alpha > 1$. Also since $f(0) = f(1) = 0$, and $f'(0) < 0, f'(1) > 0$ we find that $f(\alpha) \geqslant 0$ when $\alpha \leqslant 0$ and $\alpha \quad 1$ and $f(\alpha) \leqslant 0$ when $0 \leqslant \alpha \leqslant 1$, so that

$$\left.\begin{array}{l} \dfrac{1}{\alpha - 1}\left[\sum_{i=1}^{n} x_i^{\alpha}\ y_i^{1-\alpha} - 1\right] \geqslant 0 \text{ when } \alpha > 0 \\ \qquad\qquad\qquad\qquad \leqslant 0 \text{ when } \alpha < 0 \end{array}\right]. \qquad \ldots(29)$$

This completes the proof of Renyi's inequality.

Incidentally we have established another generalised Shannon's inequality, viz.

$$\left.\begin{array}{l} \sum_{i=1}^{n} x_i^{\alpha} \quad y_i^{1-\alpha} \ln \dfrac{x_i}{y_i} \geqslant 0 \text{ when } \alpha \geqslant 1 \\ \qquad\qquad\qquad \leqslant 0 \text{ when } -\infty < \alpha \leqslant 0 \end{array}\right] \qquad \ldots(30)$$

where the equality sign holds iff only $x_i = y_i$ for all i.

5. Some Convex Semi-Metric Spaces

(a) The ordinary n-dimensional Euclidean space (X, d) is a convex metric space since it satisfies all the postulates of a metric space and since

$$d(X, Y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2} \qquad \ldots(31)$$

is a convex function of both $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$.

(b) We give some more examples of convex semi-metric spaces based on the results of Burbea and Rao[4]. Let φ (.) be a function defined on the open interval (0.1), then we define

$$H_{n,\varphi}\ (x, y) = \sum_{i=1}^{n} \left\{ \tfrac{1}{2}\left[\varphi(x_i) + \varphi(y_i)\right] - \varphi\left(\frac{x_i + y_i}{2}\right)\right\} \qquad \ldots(32)$$

$$K_{n,\varphi}\ (x, y) = \sum_{i=1}^{n} (x_i - y_i)\left[\frac{\varphi(x_i)}{x_i} - \frac{\varphi(y_i)}{y_i}\right] \qquad \ldots(33)$$

$$L_{n,\varphi}\ (x, y) = \sum_{i=1}^{n} \left[x_i\, \varphi\left(\frac{y_i}{x_i}\right) + y_i\, \varphi\left(\frac{x_i}{y_i}\right)\right]. \qquad \ldots(34)$$

$$\text{If } \varphi(x) = (\alpha - 1)^{-1}\,(x^{\alpha} - x),\ \alpha \neq 1$$
$$= x \ln x,\ \alpha = 1 \qquad \ldots(35)$$

then we get

$$\left.\begin{aligned} J_{n,\alpha}(x, y) &= (\alpha - 1)^{-1} \sum_{i=1}^{n} \left\{\tfrac{1}{2}\left(x_i^{\alpha} + y_i^{\alpha}\right) - [(x_i + y_i)/2]^{\alpha}\right\},\ \alpha \neq 1 \\ &= \tfrac{1}{2} \sum_{i=1}^{n} \{x_i \ln x_i + y_i \ln y_i - (x_i + y_i) \ln [(x_i + y_i)/2],\ \alpha = 1 \end{aligned}\right\} \qquad \ldots(36)$$

$$\left.\begin{aligned} K_{n,\alpha}(x, y) &= (\alpha - 1)^{-1} \sum_{i=1}^{n} (x_i - y_i)\left(x_i^{\alpha-1} - y_i^{\alpha-1}\right),\ \alpha \neq 1 \\ &= \sum_{i=1}^{n} (x_i - y_i)(\ln x_i - \ln y_i),\ \alpha = 1. \end{aligned}\right\} \qquad \ldots(37)$$

$$\left.\begin{aligned} L_{n,\alpha}(x, y) &= (\alpha - 1)^{-1} \left[\sum_{i=1}^{n}\left(x_i^{\alpha}\ y_i^{1-\alpha} + \sum_{i=1}^{n} x_i^{1-\alpha} y_i^{\alpha-2}\right)\right],\ \alpha \neq 1 \\ &= \sum_{i=1}^{n} (x_i - y_i)(\ln x_i - \ln y_i),\ \alpha = 1. \end{aligned}\right\} \qquad \ldots(38)$$

We then have the following results :

(i) $[X, J_{n,\varphi}(x, y)]$ is a convex semi-metric space if φ is a C^2 convex function and $(\varphi'')^{-1}$ is concave on (0, 1)

(ii) $[X, J_{n,\alpha}(x, y)]$ is convex semi-metric space if $\alpha \in [1, 2]$ for $n = 2$. For $n = 2$ however this property is also true for $\alpha \in [3, 11/3]$

(iii) $[X, K_{n,\alpha}(x, y)]$ is convex semi-metric space for $x_i > 0, y_i > 0$ if $\alpha \in [1,2]$,

(iv) $[X, L_{n,\alpha}(x, y)]$ is convex semi-metric space for $x_i > 0,\ y_i > 0$ if $f(t) = t\varphi(t^{-1}) + \varphi(t)$ is nonegative and convex for $t > 0$.

(v) $[X, L_{n,\alpha}(x, y)]$ is a convex semi-metric space for $x_i > 0, y_i > 0$ for all $\alpha > 0$.

(c) Still another example is provided by recent generalisation of Behara and Naths'[2] entropy by Kapur[9,10] to get a measure of directed divergence viz.

$$K_{\alpha,\beta}(x, y) = (\alpha - \beta)^{-1}\left[\sum_{i=1}^{n} x_i^{\alpha}\ y_i^{1-\alpha} - \sum_{i=1}^{n} x_i^{\beta}\ y_i^{1-\beta}\right] \qquad \ldots(39)$$

and its symmetrical version

$$\bar{K}_{\alpha,\beta}(x, y) = K_{\alpha,\beta}(x, y) + K_{\beta,\alpha}(x, y). \qquad \ldots(40)$$

Both (39) and (40) are valid when

$$x_i > 0,\ y_i > 0,\ (i = 1, 2, \ldots, n),\ \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i;\ \alpha > 1,\ 0 < \beta < 1 \text{ or}$$

$$0 < \alpha < 1;\ \beta > 1. \qquad \ldots(41)$$

(d) Another interesting example is given by Kapur's[11,12] generalisation of Kullback and Leibler's[12] measure of directed divergence, viz.

$$I_a(x, y) = \sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} - \frac{1}{a}\sum_{i=1}^{n}(1 + ax_i)\ln \frac{1 + ax_i}{1 + ay_i}. \qquad \ldots(42)$$

Since

$$\mathop{Lt}_{a\to 0} \frac{\sum_{i=1}^{n}(1 + ax_i)\ln(1 + ax_i) - \sum_{i=1}^{n}(1 + ax_i)\ln(1 + ay_i)}{a}$$

$$= \sum_{i=1}^{n}(x_i - y_i) = 0 \qquad \ldots(43)$$

$I_a(x, y)$ approaches Kullback-Leibler measure of directed divergence as $a \to 0$.

The minimum value of $I_a(x, y)$ regarded as a function of x, subject to $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ is obtained when $x_i = y_i$ for all i and this minimum value is zero. This gives another generalisation of Shannon's inequality viz. that

$$\sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} - \frac{1}{a} \sum_{i=1}^{n} (1 + ax_i) \ln \frac{1 + ax_i}{1 + ay_i} \geqslant 0 \qquad \text{...(44)}$$

when $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. This inequality is valid for all values of $a \geqslant 0$ and so represents an infinity of inequalities. Shannon's inequality shows that if x_i and y_i are not all equal and $a > 0$

$$\sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} > 0, \quad \frac{1}{a} \sum_{i=1}^{n} (1 + ax_i) \ln \frac{1 + ax_i}{1 + ay_i} > 0. \qquad \text{...(45)}$$

However (44) gives a stronger result than (45) since it shows that the first positive value is greater than the second, whatever be the positive value of a.

Now (X, I_a) is not in general a convex semi-metric space, since $I_a(x, y)$ is not a convex function of y. However let us consider

$$J_a(x, y) = I_a(x, y) + I_a(y, x) \qquad \text{...(46)}$$

then

$$\frac{\partial^2 J_a(x, y)}{\partial y_i^2} = \frac{1}{(1 + ay_i)^2\, y_i^2} [2ax_i y_i + x_i + y_i] > 0. \qquad \text{...(47)}$$

$\therefore$ $(X, J_a(x, y))$ is a symmetric convex semi-metric space.

(e) A fifth example is given by

$$\bar{I}_a(x, y) = \sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} - \frac{1}{a^2} \sum_{i=1}^{n} (1 + ax_i) \ln \left(\frac{1 + ax_i}{1 + ay_i} \right) \qquad \text{...(48)}$$

$$\underset{a \to 0}{Lt}\ \bar{I}_a(x,y) = \sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} - \tfrac{1}{2} \sum_{i=1}^{n} (x_i - y_i)^2 \qquad \text{...(49)}$$

$$\frac{\partial^2 \bar{I}_a}{x\partial_i^2} = \frac{1 - x_i + ax_i}{x_i(1 + ax_i)}, \quad \frac{\partial^2 \bar{I}_a}{\partial y_i^2} = \frac{x_i}{y_i^2} - \frac{1 + ax_i}{(1 + ay_i)^2};$$

$$\frac{\partial^2 \bar{I}_a}{\partial x_i\, \partial x_j} = 0, \quad \frac{\partial^2 \bar{I}_a}{\partial y_i\, \partial y_j} = 0. \qquad \text{...(50)}$$

Let $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ be probability distributions then $\bar{I}_a(x, y)$ is a convex function of x, but is not necessarily a convex function of y. Minimizing $\bar{I}_a(x, y)$ with respect to x subject to $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1$, we find that the minimum value occurs when $x_i = y_i$ for all i and the minimum value is zero, so that we get the inequality

$$\sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} \geqslant \frac{1}{a^2} \sum_{i=1}^{n} (1 + ax_i) \ln \frac{1 + ax_i}{1 + ay_i} \qquad \ldots(51)$$

for all $a > 0$. This is another generalisation of Shannon's inequality and in fact it gives an infinity of inequalities. In particular it gives

$$\sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} - \tfrac{1}{2} \sum_{i=1}^{n} (x_i - y_i)^2 \geqslant 0 \qquad \ldots(52)$$

when

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1.$$

This can be proved independently by showing that the LHS of (52) is a convex func- of $x_1, x_2, \ldots, x_n$ which attains its minimum value zero when $x_i = y_i$ for all i.

If $a \geqslant 1$, then $I_a > 0 \Rightarrow \bar{I}_a > 0$, but if $a < 1$, then $\bar{I}_a > 0 \not\Rightarrow I_a > 0$.

If $a \leqslant 1$, then $\bar{I}_a > 0 \Rightarrow \bar{I}_a > 0$ but if $a > 1$, then $\bar{I}_a > 0 \not\Rightarrow I_a > 0$.

Thus (44) and (51) are independent inequalities, though for some values af a, one can be deduced from the other.

Again let

$$\bar{J}_a(x, y) = \bar{I}_a(x, y) + \bar{I}_a(y, x) \qquad \ldots(53)$$

then

$$\frac{\partial^2 \bar{J}_a(x, y)}{\partial x_i^2} = \frac{1 - x_i + ax_i}{x_i(1 + ax_i)} + \frac{y_i}{x_i^2} - \frac{1 + ay_i}{a(1 + ax_i)^2}$$

$$= \frac{x_i + y_i^2 + 2ax_i y_i + (a-1)\left[ax_i^3 + \frac{2a+1}{a} x_i^2 + (a+1)x_i^2 y_i\right]}{x_i^2 (1 + ax_i)^2} \qquad \ldots(54)$$

Therefore if $a > 1$, $\bar{J_a}(x, y)$ is a convex function of x and since it is asymmetrical function of x and y, it is also a convex function of y. Thus (X, J_a) is a symmetric convex semi-metric space when $a > 1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1$.

6. Comparison of Basic Postulates for Convex Semi-Metric Spaces and Normed Linear Spaces

Theorem 1—Every normed linear space gives rise to a convex semi-metric space, but a convex semi-metric space need not give rise to a normed linear space.

Proof : We can use the norm $\|x\|$ of the normed linear space X to define $d(x, y) = \|x - y\|$ and show that (X, d) is a metric space and $d(x, y)$ is a convex function of both x and y so that every normed linear space gives rise to a convex metric space. However our examples show that the converse is not true.

We also see that if $0 \leqslant \lambda \leqslant 1$ and $d(x, y)$ is a convex function of y, then

$$d(x, (1 - \lambda)x + \lambda y) \leqslant (1-\lambda)\, d(x, x) + \lambda d(x, y) = \lambda\, d(x, y) \quad \text{.. (55)}$$

while if $d(x, y) = \|x - y\|$, then

$$d(x, (1 - \lambda)x + \lambda y) = \|x - (1 - \lambda)x - \lambda y\| = \|\lambda x - \lambda y\| = \lambda \|x - y\|. \quad \text{...(56)}$$

Now (55) $\Rightarrow$ (56) but (56) does not imply (55), so that normed linear space structure implies convexity, but convexity in a metric space does not imply normed linear space structure. In Fig. 1, convexity requires $PR \leqslant \lambda PQ$, while normed linear

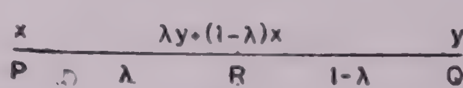

Fig. 1.

space structure requires $PR = \lambda PQ$. Figure 2 illustrate the relation between the various spaces.

We have the following results :

(a) Every normed linear space has a corresponding metric space, but the converse is not true.

(b) Every convex metric space is a metric space, but the converse is not true.

(c) Every convex metric space satisfies the conditions for a convex symmetric semi-metric space, but the converse is not true.

(d) Every convex symmetric semi-metric space satisfies the condition for a convex semi-metric space, but the converse is not true.

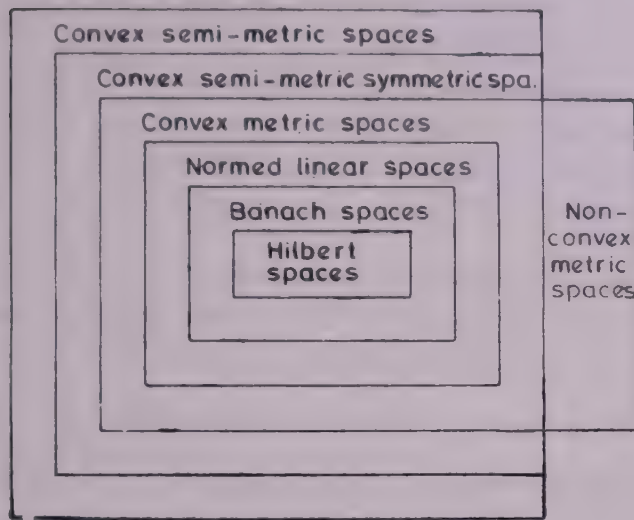

FIG. 2. Relations between various spaces.

From normed linear space, we can obtain smaller classes of Banach and Hilbert spaces by imposing additional conditions of completeness and inner product being defined. On the other hand, we can obtain bigger classes of symmetric and non-symmetric convex semi-metric spaces by relaxing postulates (iii) and (iv) of section 1 and replacing these by the postulate of convexity. Since convexity is implied by (iii) and (iv), we are retaining a part of (iii) and (iv) (and an essential part at that) and giving up the rest. In view of the importance of convexity concept for applications in mathematical programming, mathematical economics, information theory etc., this is a useful investigation. Of course, we have restricted to the space of ordered n-tuples of non-negative real numbers satisfying $\sum_{i=1}^{n} xi = k$, where k is a constant. The case $k = 1$ is important because in this case, the ordered n-tuples represent probability distributions.

From an alternative point of view, we are trying to abstract some properties of distance between x and y. Firstly we are talking of distance of x from y and not of distance between x and y and as such we give up the requirement of symmetry. However we have insisted that distance of x from x should be zero and distance of x from another point y distinct from x is positive. The only additional property used is convexity which implies that (i) distance of x from $\lambda y + (1 - \lambda) x \leqslant \lambda$ times the distance of x from y and (ii) distance of x from $\lambda y_1 + (1 - \lambda) y_2 \leqslant \lambda$ times the distance of x from y_1 + $(1 - \lambda)$ times the distance of x from y_2. For normed linear spaces, we require in (i) and (ii) equality rather than inequality. The replacement of equality by inequality here extends the class of normed linear spaces to convex semi-metric spaces.

However, as noted earlier, in constructing our examples, we restricted ourselves to ordered n-tuples of non-negative or positive real numbers satisfying that the sum

of the components is always a constant i. e. instead of considering all points in n-dimensional space, we considered only all points in a hyperplane in the positive orthant. However since probability distributions represent such ordered n-tuples in space, the semi-metrics, introduced are useful in discussing divergence between probability distributions. Moreover this discussion can be extended to give measures of directed divergence between probability distribution of continuous variates.

Incidentally this approach gives a number of generalisations of Shannon's inequality e. g. (29), (30), (44) and (51). In each case there is a corresponding inequality for continuous variate probability density functions.

7. Generation of Convex Semi-Metric Spaces

Given a number of convex semi-metric spaces on the same set X, we can find more convex semi-metric spaces by using the following theorems.

Theorem 2—If $(X, d_1(x, y)), (X, d_2(x, y)), \ldots, (X, d_m(x, y))$ are convex semi-metric spaces, then so is

$$(X, \sum_{j=1}^{m} a_j\, d_j(x, y))$$

where $a_j \geqslant 0$ for $j = 1, 2, \ldots\ m$.

Proof : Let $d(x, y) = \sum_{j=1}^{m} a_j\, d_j\ (x, y)$, then it is easily seen that $d(x, y) \geqslant 0$, $d(x, y) = 0$ iff $x = y$ and $d(x, y)$ is a convex function of both x and y.

Theorem 3— Let $g(x)$ be a non-negative convex increasing function of x when $x \geqslant 0$, which vanishes only when $x = 0$, then $(X, d(x, y))$ is a metric space $\Rightarrow$ $(X, g(d(x, y))$ is a convex semi-metric space.

Proof : This follows from the results that (i) a convex increasing function of a convex function is a convex function, (ii) $g(d(x, y)) > 0$ (iii) $g(d(x, y)) = 0$ iff $d(x, y) = 0$, and (iv) $d(x, y) = 0$ iff $x = y$.

Theorem 4—If $(X, d_j(x, y)), (j = 1, 2, \ldots, m)$ are convex semi-metric spaces, then so is $(X, \sum_{j=1}^{m} a_j\, d_j^{p_j}(x, y))$ wher $a_j > 0, p_j > 1$ for $j = 1, 2, \ldots, m$.

Proof : This follows from Theorems 1 and 2 on using the result that x^p, when $p > 1$, is a convex increasing function of x.

Thus we can generate an infinity of convex semi-metric spaces. Most of these arise when X is the set of ordered n-tuples of positive real numbers $(x_1, x_2, \ldots, x_n)$ where $\sum_{i=1}^{n} x_i = k$. Most of these can again be extended to the case when each $x_i \geqslant 0$.

Almost all can be extended to the case of continuous density functions and give generalised Shannon or Shannon-type inequalities.

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AN ANALOGUE OF BISHOP'S THEOREM FOR REAL FUNCTION ALGEBRAS

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In this paper analogues of Bishop's and Machado's theorems for a real function algebra are proved.

1. INTRODUCTION

Burckel[2] has clearly pointed out the importance of the Stone-Weierstrass theorem in modern abstract analysis. This theorem has undergone many generalizations as pointed out in Burckel[2]. One of the well-known generalizations is the theorem due to Bishop[1], which enables one to reduce the study of arbitrary function algebras to the case of anti-symmetric algebras.

Let X be a compact Hausdorff Space, $C(X)$ the Banach algebra of all continuous complex valued functions with the supremum norm and A a unital subalgebra of $C(X)$. A non-empty subset K of X is said to be A-antisymmetric, if $h \in A$ and h is real valued on K implies that h is constant on K.

Bishop's Theorem[1]

Suppose A is a closed unital subalgebra of $C(X)$ and $f \in C(X)$. If for each maximal A-antisymmetric subset K of X, there exists $g \in A$ such that $f|_K = g|_K$, then $f \in A$.

It follows that if A is a function algebra (that is a closed unital subalgebra of $C(X)$ with the property that for $x \neq y \in X$, there exists $f \in A$ such that $f(x) \neq f(y)$) and if $\bar{f} \in A$ for each $f \in A$, then every A-antisymmetric set is singleton. Hence every $f \in C(X)$ satisfies the hypotheses of Bishop's theorem and therefore belongs to A. Thus $A = C(X)$, which is the Stone-Weierstrass Theorem.

The classical proofs of Bishop's Theorem made use of the non-trivial tools in functional Analysis like Hahn-Banach theorem, Riesz-Representation theorem, Krein-Mil'man theorem etc. (see Burckel[2] for further references to these proofs). Machado[4] formulated a more general and quantitative version of Bishop's theorem and gave a completely elementary and elegant proof of it.

Let $f \in C(X)$ and K be a non-empty closed subset of X. We define

$$\|f\|_K = \text{Sup}\{\,|f(x)| : x \in K\}.$$

Machado's Theorem[4]

Let $f \in C(X)$. Then there exists a closed A-antisymmetric set S such that

$$\inf\{\|f - g\|_S : g \in A\} = \inf\{\|f - g\|_X : g \in A\}.$$

Machado's proof is presented in a self-contained manner in Burckel[2]. This proof is further simplified and shortened by Ransford[5].

The aim of the present note is to prove analogues of Bishop's and Machado's theorems for a real function algebra. (Definition 2.1). Real function algebras were defined in Kulkarni and Limaye[3] as follows : Let X be a compact Hausdorff space and $\tau : X \to X$, a homeomorphism such that $\tau \circ \tau =$ Identity on X.

Let

$$C(X, \tau) = \{f \in C(X) : f(\tau(x)) = \bar{f}(x) \text{ for all } x \in X\}.$$

A closed (real) subalgebra A which contains (real) constants and separates the points of X is called a real function algebra. A non-empty subset K of X is called an A-anti-symmetric set if $f \in A$ and f is purely real or purely imaginary on K implies that f is constant on K. (Note the difference between this definition and that of an antisymmetric set with respect to a complex function algebra See remark 2.3). We show that with this definition of an anti-symmetric set, Machado's theorem remains valid for a real function algebra (Theorems 3.3). Many applications of Bishop's theorem for complex function algebras depend on the fact that if A is a is a complex function algebra and K is an A-antisymmetric set, then $A \mid_K$ is closed in $C(K)$. We show that this fact is also true in case of real function algebras (Theorem 3.6).

The Second Section gives the definitions and elementary properties of anti-symmetic sets. These properties are used to prove the main results in the last Section.

2. Anti-symmetric Sets

Let X be a compact-Hausdorff space and $C(X)$, the Banach algebra of all continuous complex-valued functions on X, with the supremum norm.

Let $\tau : X \to X$ be a homeomorphism such that $\tau^2 = \tau \circ \tau$ is the identity map on X.

Let

$$C(X, \tau) = \{f \in C(X) : f(\tau(x)) = \bar{f}(x) \text{ for all } x \in X\}.$$

Then $C(X, \tau)$ is a real commutative Banach algebra with the identity 1. Also it is easy to see that $C(X, \tau)$ separates the points of X, that is, for any $x_1 \neq x_2$ in X, there is f in $C(X, \tau)$ such that $f(x_1) \neq f(x_2)$.

Definition 2.1—Let X be a compact-Hausdorff space and $\tau : X \to X$ a homeomorphism such that $\tau^2 = \tau \circ \tau$ is the identity map on X. A real function algebra on (X, τ) is a (real) subalgebra A of $C(X, \tau)$ that

(i) is uniformly closed,

(ii) contains real constants, and

(iii) separates the points of X.

For elementary properties of and examples of real function algebras, we refer to Kulkarni and Limaye[3].

Definition 2.2—Let A be a real function algebra on (X, τ). A non-empty subset K of X is called a set of anti-symmetry of A or an A-antisymmetric set if

(i) $f \in A$ and $f \mid_K$ is real implies $f \mid_K$ is constant, and

(ii) $f \in A$ and $f \mid_K$ is purely imaginary implies $f \mid_K$ is constant

A is said to be an anti-symmetric algebra if X is an A-antisymmetric set.

Remark 2.3 : Note that in case of complex algebras conditions (i) and (ii) of Definition 2.2 are equivalent.

Now throughout the remaining part of this section, we fix a real function algebra A on (X, τ). The following properties of A-antisymmetric sets are elementary.

Lemma 2.4—If K and H are A-antisymmetric sets and $K \cap H \neq \phi$, then $K \cup H$ is A-antisymmetric.

PROOF : Let $x \in K \cap H$ and $f \mid_{(K \cup H)}$ be real. Then obviously

$$f \mid_K = \{f(x)\} = f \mid_H .$$

Hence

$$f \mid_{(K \cup H)} = \{f(x)\}.$$

Similar proof if $f \mid_{(K \cup H)}$ is purely imaginary.

Lemma 2.5—Let $\phi \neq Y$ be an A-antisymmetric set and $\{K_\lambda : \lambda \in \Lambda\}$ be a family of A-antisymmetric sets such that $Y \subset K_\lambda$ for each $\lambda \in \Lambda$. Then $K = \bigcup_{\lambda \in \Lambda} K_\lambda$ is also A-antisymmetric.

PROOF : Same as proof of Lemma 2.4, which is in fact a special case of this lemma.

Remark 2.6 : It follows from the above Lemma and from Zorn's Lemma, that every A-antisymmetric set is contained in a maximal A-antisymmetric set. In parti-

cular, since $\{x\}$ is trivially A-antisymmetric, each $x \in X$ lies in a maximal A-antisymmetric set.

Lemma 2.7—If K is an A-antisymmetric set, then $\bar{K}$ (the closure of K) is also A-antisymmetric.

PROOF: Let $f \in A$ be real on $\bar{K}$. Then f is real on K and hence constant. Hence by continuity of $f, f \mid_{\bar{K}}$ is constant. Similar proof if f is purely imaginary on $\bar{K}$.

Corollary 2.8—Let K be a maximal A-antisymmetric set. Then K is closed.

PROOF: By Lemma 2.7 $\bar{K}$ is A-antisymmetric and hence by maximality of K, $\bar{K} \subset K$.

Lemma 2.9—Distinct maximal A-antisymmetric sets must be disjoint.

PROOF: Follows from Lemma 2.4.

The properties of A-antisymmetric sets discussed so far are analogues to the corresponding properties of antisymmetric sets of a complex function algebra. (see Burckel[2] for these properties). Now we describe a few properties that are particular to real function algebras.

Lemma 2.10—Let K be an A-antisymmetric set. Then $\tau(K)$ is also A-antisymmetric.

PROOF: Let $f \in A$ be real (respectively purely imaginary) on $\tau(K)$. Then since $f(x) = \bar{f}(\tau(x))$ for all x, we have $f \mid_K$ is real (respectively purely imaginary) and hence constant say C. But then $f \mid_{\tau(K)} = \bar{C}$.

Corollary 2.11—Let K be a maximal A-antisymmetric set. Then either $K = \tau(K)$ or $K \cap \tau(K) = \phi$.

PROOF: It follows from Lemma 2.10 that $\tau(K)$ is A-antisymmetric. If $\tau(K) \subset H$ and H is A-antisymmetric, then $\tau(H)$ is A-antisymmetric by Lemma 2.10. But $K \subset \tau(H)$. Hence $K = \tau(H)$ by maximality of K. But then $\tau(K) = H$. This implies that $\tau(K)$ is also a maximal A-antisymmetric set. Now the conclusion follows from Lemma 2.9.

Now we define

$$B = \{f + ig : f, g \in A\}.$$

It can be shown that B is a complex function algebra on X and can be regarded as the complexification of A. (see Kulkarni and Limaye[8] for proof). Now we investigate the relationship between A-antisymmetric sets and B-antisymmetric sets. Note that in view of Remark 2.3, to show that a non-empty subset K of X is B-antisymmetric, it is sufficient to prove that $f \in B$ and $f \mid_K$ is real implies $f \mid_K$ is constant.

The following Lemma is obvious.

Lemma 2.12—Let K be a B-antisymmetric set. Then K is A-antisymmetric.

We do not know whether the converse of the above Lemma is also true. In what follows, we show that under some additional conditions, an A-antisymmetric set is also B-antisymmetric.

Lemma 2.13—Let K be a B-antisymmetric set. Then $\tau(K)$ is also B-antisymmetric.

PROOF : Note that for $x \in X$,

$$(f - ig)(\tau(x)) = \bar{f}(x) - i\bar{g}(x) = \overline{(f + ig)(x)}\,.$$

Thus $f + ig$ is real (respectively constant) on K, iff $f - ig$ is real (respectively constant) on $\tau(K)$. From this the Lemma follows.

Corollary 2.14—Let K be a maximal B-antisymmetric set. Then either $K = \tau(K)$ or $K \cap \tau(K) = \phi$.

PROOF : Follows from the fact that distinct maximal B-antisymmetric sets are disjoint.

Theorem 2.15—Let K be an A-antisymmetric set. Such that $\tau(K) = K$. Then K is B-antisymmetric.

PROOF : Let $f + ig \in B$ and $(f + ig)\mid_K$ be real. Then as noted in the proof of Lemma 2.13.

$$(f - ig)(\tau(x)) = \overline{(f + ig)(x)} \text{ for all } x \in X.$$

Hence $(f - ig)$ is real on $\tau(K) = K$. Thus

$$f = \tfrac{1}{2}[(f + ig) + (f - ig)] \text{ is real on } K$$

and

$$g = \frac{1}{2i}[(f + ig) + (f - ig)] \text{ is purely imaginary on } K.$$

Now since $f, g \in A$ and K is A-antisymmetric, $f\mid_K$, $g\mid_K$ are constants.

Hence $f + ig$ is constant on K.

Corollary 2.16—A is an antisymmetric algebra, if and only if B is an antisymmetric algebra.

PROOF : Follows from Lemma 2.12, Theorem 2.15 and the fact that $X = \tau(X)$.

Theorem 2.17—Let K be a maximal A-antisymmetric set. Then K is B-antisymmetric.

PROOF : By corollary 2.11.

$$K = \tau(K) \text{ or } K \cap \tau(K) = \phi.$$

If $K = \tau(K)$, then by Theorem 2.15, K is B-antisymmetric. Now suppose $K \cap \tau(K) = \phi$. Since $K \cup \tau(K)$ is not A-antisymmetric, there is $h \in A$ such that h is real or purely imaginary on $K \cup \tau(K)$ and non constant on $K \cup \tau(K)$. But $h \mid_K$ as well as $h \mid_{\tau(K)}$ must be constant. Since $h(\tau(x)) = \bar{h}(x)$ for all $x \in K \subset X$, $h \mid_K$ cannot be real. Hence $h \mid_K$ is purely imaginary and non zero constant. We may assume that $h \mid_K = i$. Now let $(f + ig) \mid_K$ be real. We have $(f + ig) \mid_K = (f + hg) \mid_K =$ constant since $f + hg \in A$. Thus K is B-antisymmetric.

Corollary 2.18 – K is a maximal A-antisymmetric set iff K is a maximal B-antisymmetric set.

PROOF : Let K be a maximal A-antisymmetric. Then by Theorem 2.17, K is B-antisymmetric. If $K \subset H$ and H is B-antisymmetric, then by Lemma 2.12, H is A-antisymmetric. Hence $K = H$. Thus K is a maximal B-antisymmetric set.

Now suppose that K is a maximal B-antisymmetric set. By Lemma 2.12, K is A-antisymmetric. Hence by remark 2.6, K is contained in a maximal A-antisymmetric set T. But then by Theorem 2.17 T is B-antisymmetric and hence $K = T$, proving that K is a maximal A-antisymmetric set.

3. THEROEMS OF BISHOP AND MACHADO

We present the proofs of main theorems in this Section using the properties of antisymmetric sets discussed in the last Section. We use the following notation throughout this Section.

Definition 3.1—Let X be a compact Hausdorff space. For $f \in C(X)$ and K, a non empty closed subset of X define

$$\|f\|_K = \text{Sup}\,\{\,|f(x)| : x \in K\}.$$

If A is a subalgebra (real or complex) of $C(X)$ define

$$d_K(f, A) = \inf\,\{\|f - g\|_K : g \in A\}.$$

Remark 3.2 : It is to see that in the above definition if H is a closed subset of X such that $K \subset H$,

then

$$d_K(f, A) \leqslant d_H(f, A).$$

Also if A is a subalgebra of B then

$$d_K(f, A) \geqslant d_K(f, B).$$

Theorem 3.3 (*Machado's Theorem*)—Let X be a compact Hausdorff space, τ a homeomorphism on X such that $\tau \circ \tau =$ identity map on X. Let A be a real function algebra on (X, τ) and $f \in C(X, \tau)$. Then there exists a non-empty closed A-anti-symmetric subset Y of X such that

$$d_Y(f, A) = d_X(f, A).$$

Proof : This proof is a slight modification of Ransford's proof[5].

Let Λ denote the collection of all non-empty closed subsets F of X such that $\tau(F) = F$ and $d_F(f, A) = d_X(f, A)$. Clearly $X \in \Lambda$, thus Λ is non empty. Let $\mathscr{H}$ be a totally ordered (by inclusion) subfamily of Λ, and let

$$E = \{\cap F : F \in \mathscr{H}\}.$$

Claim 1—$E \in \mathscr{H}$: Clearly E is closed and $\tau(E) = E$. Also since each $F \in \mathscr{H}$ is a non empty compact set and E is the intersection of the chain of such sets E is also non- empty. Thus we have only to prove that

$$d_E(f, A) = d_X(f, A).$$

Now for $g \in A$, $F \in \mathscr{H}$ define

$$S(F) = \{x \in F : |f(x) - g(x)| \geq d_X(f, A)\}.$$

Then each $S(F)$ is a non empty compact set. Hence the intersection of the chain of all such sets namely,

$$\cap \{S(F) : F \in \mathscr{H}\} = \{x \in E : |f(x) - g(x)| \geq d_X(f, A)\}$$

is non empty.

Hence

$$\|f - g\|_E \geqslant d_X(f, A) \text{ for all } g \in A.$$

It follows that $d_E(f, A) = d_X(f, A)$. This proves the claim. Now by Zorn's Lemma there exists a minimal element S in Λ.

Claim 2—If $g \in A$ and $g \mid S$ is real then $g \mid S$ is constant. Suppose there exists $g \in A$ such that $g \mid S$ is real and nonconstant. We may assume that

$$\min_{x \in S} g(x) = 0 \text{ and } \max_{x \in S} g(x) = 1.$$

Let

$$B = \{x \in S : 0 \leqslant g(x) \leqslant 2/3\}$$

$$C = \{x \in S : 1/3 \leqslant g(x) \leqslant 1\}.$$

Then B and C are non-empty, closed, proper subsets of S. It is easy to see that $\tau(B) = B$ and $\tau(C) = C$. Hence the minimality of S implies the existence of two elements g_B and g_C in A such that

$$\|f - g_B\|_B < d_X(f, A)$$

$$\|f - g_C\|_C < d_X(f, A). \qquad \ldots(1)$$

Now for $n \geqslant 1$ define $g_n = (1 - g^n)^{2^n}$ and

$$P_n = g_n g_B + (1 - g_n) g_C.$$

Thus $g_n, P_n \in A$ and $0 \leqslant g_n \leqslant 1$ on S for all n.
Since P_n is a convex combination of g_B and g_C it follows from (1) that

$$\|f - p_n\|_{B \cap C} < d_X(f, A). \qquad \ldots(2)$$

On $B - C, 0 \leqslant g < 1/3$ hence

$$g_n = (1 - g^n)^{2^n} \geqslant 1 - 2^n g^n \geqslant 1 - (2/3)^n.$$

whereas on $C - B$ since $2/3 < g \leqslant 1$, we have

$$g_n = (1 - g^n)^{2^n} \leqslant (1 + g^n)^{-2^n} \leqslant (2^n g^n)^{-1} \leqslant (3/4)^n.$$

Thus p_n converges uniformly to g_B on $B - C$ and to g_C on $C - B$.

Combining this with (1) and (2) we see that for large n,

$$\|f - p_n\|_S < d_X(f, A)$$

that is, $d_S(f, A) < d_X(f, A)$, contradicting the fact that $S \in \Lambda$. This proves the claim.

Now if $h \in A$ and $h \mid S$ is purely imaginary implies that $h \mid S$ is constant, then S is an A-antisymmetric set and we can take $Y = S$.

If not, there exists $h \in A$ such that $h \mid S$ is purely imaginary and non-constant on S. But then $h^2 \mid S$ is real and hence constant on S by Claim 2. We may assume $h^2 \mid S = -1$, that is $h(x) = \pm i$ for all $x \in S$.

Let

$$Y = \{x \in S : h(x) = +i\}$$

and

$$Z = \{x \in S : h(x) = -i\}.$$

Then $S = Y \cup Z$ and $Y \cap Z = \phi$. Since $\tau(S) = S$, $\tau(Y) \subset S = Y \cup Z$. But it is easy to see that $Y \cap \tau(Y) = \phi$, hence $\tau(Y) \subset Z$.

Similarly $\tau(Z) \subset Y$.

Thus $\tau(Y) = Z$ and $\tau(Z) = Y$.

Now we show that Y satisfies all the required properties. Clearly Y is non empty and closed. We first show that Y is A-antisymmetric.

If $g \in A$ and $g \mid_Y$ is real then $g \mid_Z$ is also real. Hence $g \mid (Y \cup Z) \mid = g \mid S$ is real. Thus g is constant on S and hence on Y. Next let $g \mid_Y$ be purely imaginary. Let $g' = gh \in A$.

We see that on Y, $g' = ig$ is real.

Hence $g' = ig$ is constant on Y that is g is constant on Y. Thus Y is A-antisymmetric.

It remains to show that $d_Y(f, A) = d_X(f, A)$. Fix $g \in A$. There exists $x \in S$ such that

$$|f(x) - g(x)| \geqslant d_X(f, A).$$

Also

$$|f(\tau(x)) - g(\tau(x))| = |\bar{f}(x) - \bar{g}(x)| \geqslant d_X(f, A).$$

Since

$$S = Y \cup Z = Y \cup \tau(Y) \text{ we have } x \in Y \text{ or } \tau(x) \in Y.$$

Thus

$$\|f - g\|_Y \geqslant d_X(f, A).$$

Since this is true for all g in A, we have

$$d_Y(f, A) \geqslant d_X(f, A).$$

This completes the proof of the theorem.

Corollary 3.4 (*Bishop's Theorem*)—Let A be a real function algebra on (X, τ). Suppose $f \in C(X, \tau)$ is such that $f \mid_K \in A \mid_K$ for every maximal-A-antisymmetric subset K of X, then $f \in A$.

PROOF : By Theorem 3.3 there is a nonempty closed A-antisymmetric subset H of X such that

$$d_H(f, A) = d_X(f, A).$$

By Remark 2.6 H is contained in a maximal A-antisymmetric set K.

But then $d_X(f, A) = d_H(f, A) \leqslant d_K(f, A) = 0$

as $f \mid_K \in A \mid_K$

Hence $f \in A$.

Corollary 3.5 (*Stone-Weierstrass*)—Let A be a real function algebra on (X, τ), such that $\bar{f} \in A$ for every $f \in A$. Then,

$$A = C(X, \tau).$$

PROOF : Let $x, y \in X$ and $x \neq y$ then there exists $f = u + iv \in A$ such that $f(x) \neq f(y)$ that is $u(x) \neq u(y)$ or $iv(x) \neq iv(y)$. But $u = \frac{f + \bar{f}}{2}$ and $iV = \frac{f - \bar{f}}{2}$ are members of A. u is real and iv is purely imaginary on $\{x, y\}$. Hence $\{x, y\}$ is not A-antisymmetric. Thus every maximal A-antisymmetric set is singleton. Hence every $f \in C(x, \tau)$ trivially satisfies the hypothesis of Corollary 3.4.

For a different proof of the above analogue of Stone-Weierstrass theorem, see Kulkarni and Limaye[3].

In order to apply the analogue of Bishop's theorem effectively, the hypotheses on A should pass over to the restriction algebra $A \mid_K$ for each maximal A-antisymmetric set K. The following theorem shows that the hypothesis of completeness can pass over in this manner. In effect this reduces the study of real function algebras to that of real-antisymmetric function algebras.

Theorem 3.6—Let A be a real function algebra on (X, τ) and K be a maximal A-antisymmetric set, then $A \mid_K$ is closed in $C(K)$.

PROOF : Let $f_n \in A$ and $f_n \mid_K \to p \in C(K)$ as $n \to \infty$. We have to show that $p \in A \mid_K$. Now let $B = \{f + ig : f, g \in A\}$ as defined in Section 2. Then B is a complex function algebra and K is a maximal B-antisymmetric set. (Theorem 2.17). Hence $p \in B \mid_K$ (see Burckel[2] for a proof).

Thus there exists $f, g \in A$ such that $(f + ig) \mid_K = p$.

Now since K is maximal A-antisymmetric, either $K = \tau(K)$ or $K \cap \tau(K) = \phi$ (Corollary 2.12).

Suppose $K = \tau(K)$.

Since $f_n \to p$ and $f_n(\tau(x)) = \bar{f_n}(x)$ for all x in K we have $p(\tau(x)) = p(x)$ for $x \in K$. But then for each $x \in K$,

$$f(x) + ig(x) = p(x) = \overline{(p(\tau(x))} = \overline{f(\tau(x)) + ig(\tau(x))}$$
$$= f(x) - ig(x).$$

Thus $g(x) = 0$ for all $x \in K$

Hence $p = f \mid_K \in A \mid_K$.

Let $K \cap \tau(K) = \phi$. Then we can prove that there exists $h \in A$ such that $h \mid_K = i$ (see proof of Theorem 2.17). But then $p = (f + hg) \mid_K \in A/K$.

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ON STRONGLY α-IRRESOLUTE MAPPINGS

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Njåstad[7] introduced the concept of α-open sets. In this paper the author introduces the concept of strongly α-irresolute ("s. α. i") mapping and presents some properties of such mappings.

1. INTRODUCTION

Let S be a subset of a topological space (X, τ). We denote the closure of S and the interior of S with respect to τ by $\bar{S}$ or $\text{cl}_X(S)$ and $\mathring{S}$ or $\text{int}_X(S)$ respectively.

A set S is called an α-open[7] (resp. semi-open: regular semi-open) set if there exists an open set A such that $A \subseteq S \subseteq \mathring{\bar{A}}$ (resp. $A \subseteq S \subseteq \bar{A}$; $\vec{A} \subseteq S \subseteq A$). The family of all α-open (resp. semi-open) sets in (X, τ) will be denoted by τ_α (resp. S. O. (X, τ)). It is known[7] that τ_α is a topology for X and $\tau_\alpha \subseteq$ S. O. (X, τ).

(X, τ) is called an α space if $\tau = \tau_\alpha$.

A mapping $f : (X,\tau) \to (Y,\sigma)$ is called irresolute[1] (resp. α-continuous[6]; α-irresolute[5]), briefly 'i.' (resp. 'α. c.'; 'α. i.') if $f^{-1}(V) \in$ S. O. (X, τ) (resp. $f^{-1}(V) \in \tau_\alpha$; $f^{-1}(V) \in \tau_\alpha$) for every $V \in$ S. O. (Y, σ) (resp. $V \in \sigma$; $V \in \sigma_\alpha$). 'α. i.' and 'i' are independent[9], and every 'α. i' mapping is 'α. c.' but a continuous mapping is not necessarily[5] 'α. i.', therefore the concept of 'α. c.' mapping is strictly weaker than that of 'α. i.' mapping. The purpose of the present note is tointroduce a strong form of continuity, called 'strongly α-irresolute', which is stronger than 'α.i.'.

2. STRONGLY α-IRRESLOUTE MAPPINGS

Definition 2.1- -A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly α-irresolute, briefly s. α. i., if for each $x \in X$ and each α-open set $L \subseteq Y$ containing $f(x)$, there exists $W \in \tau$ such that $x \in W$ and $f(W) \subseteq L$.

Remark 2.1: It is clear that if $f : (X, \tau) \to (Y, \sigma)$ is 's. α. i.' then it is 'α. i.' and continuous. But, the converse of this statement may not be true, as the following example shows.

(*) Lavoro eseguito nell'ambito del G.N.S.A.G.A. e con contributo M. P. I. (1983).

Example 2.1—Let $X = \{a, b, c, d\}$ equipped with the topology $\tau = \{\phi, X, \{a,b\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be the identity mapping. Then f is continuous, 'α. i.' but it is not 's. α. i.'.

Remark 2.2 : The concept of s. α. i. and i. are independent of each other. The following example shows that there exists a s. α. i. mapping that is not i.

Example 2.2— Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a, b, c\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$.

Define $f : (X, \tau_1) \to (X, \tau_2)$ by $f(a) = f(b) = b$, $f(c) = c$ and $f(d) = d$.

Then f is s. α. i but it is not i. because $\{a, d\} \in$ S. O. (X, τ_2) and $f^{-1}(\{a, d\}) \notin$ S. O. (X, τ_1).

The following diagram summarizes the above discussion

$$\text{“}\alpha.\text{i.”} \quad \begin{array}{c} \Leftarrow == \text{“s.}\alpha\text{.i.”} == \Rightarrow \\ \Rightarrow == \text{“}\alpha\text{.c.”} == \Leftarrow \end{array} \quad \text{continuous}$$

Theorem 2.1 is an easy consequence of Definition 2.1 and the proof is thus omitted.

Theorem 2.1—Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping, then the following statements are equivalent :

(i) f is "s. α. i";

(ii) $f : (X, \tau) \to (Y, \sigma_\alpha)$ is continuous;

(iii) the inverse image of each α-open set is open;

(iv) the inverse image of each α-closed set (the complement of an α-open set) is closed;

(v) $f(\bar{A}) \subseteq \overline{f(A)}^{\alpha}$ (where $(^{-\alpha})$ denotes the α-closure) for each $A \subseteq X$;

(vi) $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B}^{\alpha})$, for each $B \subseteq Y$;

(vii) $f^{-1}(C^{\circ\alpha})\, \overset{\circ}{\widehat{f^{-1}(C)}}$ (where $(^{\circ}\alpha)$ denotes the α-interior), for each $C \subseteq Y$;

(viii) If a filterbas is $\mathscr{F}$ of X converges to a point x in X, then the filter basis $f(\mathscr{F}) = \{f(A) : A \in \mathscr{F}\}$ α-converges to $f(x)$ in $\mathscr{Y}$ (a filter basis α-converges to $x \in X$, if and only if for each α-open set L of X containing x there exists $G \in \mathscr{G}$ such that $G \subseteq (L)$.

Lemma 2.2—If a mapping $f : (X, \tau) \to (Y, \sigma)$ is "s. α. i.", then $f^{-1}(N)$ is closed for any nowhere dense subset N of Y.

Proof : If N is a nowhere dense subset of Y, then $Y - N$ is an α-open set of Y and hence $f^{-1}(Y - N) = X - f^{-1}(N) \in \tau$. It follows that $f^{-1}(N)$ is a closed set of X.

Theorem 2.3—Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping, then the following statements are equivalent :

(i) f is 's. α. i.';

(ii) f is continuous and the inverse image of each nowhere dense set is closed.

PROOF : (i) $\Rightarrow$ (ii). It follows from Lemma 2.2.

(ii) $\Rightarrow$ (i). Let L be an α-open set of Y, then L may be written as a difference between an open set A and a nowhere dense set $N \subseteq \mathring{A}$ [Njastad[7], Prop. 4].
Since $f^{-1}(L) = f^{-1}(A - N) = f^{-1}(A) - f^{-1}(N) = f^{-1}(A) \cap (X - f^{-1}(N)) \in \tau$, the proof is complete.

Theorem 2.4—Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping, then the following statements are equivalent :

(i) f is "s. α. i.";

(ii) for each $y \in Y$ and each open set $U \subseteq Y$ such that $y \in \mathring{\bar{U}}$, the inverse image of $U \cup \{y\}$ is an open subset of X.

PROOF : (i) $\Rightarrow$ (ii). We have $U \subseteq U \cup \{y\} \subseteq \mathring{\bar{U}}$ and then $U \cup \{y\}$ is an α-open set of Y and hence $f^{-1}(U \cup \{y\}) \in \tau$.

(ii) $\Rightarrow$ (i). Let L be an α-open subset of Y. There existsan open set $U \subseteq Y$ such that $U \subseteq L \subseteq \mathring{\bar{U}}$ and then for each $y \in L$, $f^{-1}(U \cup \{y\}) \in \tau$.

Since $f^{-1}(L) = \bigcup_{y \in L} f^{-1}(U \cup \{y\}) \in \tau$, f is "s. α. i.".

Definition 2.2—Let (X, τ) and (Y, σ) be topological spaces. A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be α-open (resp. semi-open; pre-semi-open) if, for all $A \in \tau$ (resp. $A \in \tau$; $A \in$ S. O. (X, τ)) $f(A) \in \sigma_\alpha$ (resp. $f(A) \in$ S. O. (Y, σ); $f(A) \in$ S. O. (Y, σ)).

Theorem 2.5—If $f : (X, \tau) \to (Y, \sigma)$ is an α-open "s. α. i." mapping, then f is "i", and pre-semi-open.

PROOF : $f : (X, \tau) \to (Y, \sigma_\alpha)$ is an open continuous mapping and then irresolute and pre-semi-open (Grossley and Hilderband[1], Theorem 18). Since S. O. (Y, σ) = S. O. (Y, σ_α) (Faro[4], Prop. 3.3), $f : (X, \tau) \to (Y, \sigma)$ is irresolute and pre-semi-open.

Example 2.2—Shows that in Theorem 2.5 α-openess of f cannot be dropped even if f is pre-semi-open.

Theorem 2.6—The following statements are equivalent:

(i) (X, τ) is an α-space:

(ii) every continuous mapping $f : (Y, \sigma) \to (X, \tau)$ is "s. α. i.";

(iii) the identity mapping $i : (X,\tau) \to (X, \tau)$ is "s. α. i.".

PROOF : (i) $\Rightarrow$ (ii). It follows from (ii) of Theorem 2.1.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Since $i : (X, \tau) \to (X, \tau)$ is "s. α. i.", $i^{-1}(L) \in \tau$ for any $L \in \tau_\alpha$ and hence X is an α-space.

Lemma 2.7—If $f : (X, \tau) \to (Y, \sigma)$ is a closed one-to-one continuous mapping, then the image of each nowhere dense set $C \subseteq X$ is nowhere dense.

PROOF : Suppose that there exists a nowhere dense set $C \subseteq X$ such that $\overline{f(\mathring{C})} \neq \phi$ We can find $y \in Y$ and $A \in \sigma$ such that $y \in A \subseteq \overline{f(C)} = f(\overline{C})$ and so $f^{-1}(y) \in f^{-1}(A) \subseteq f^{-1}(f(\overline{C})) = \overline{C}$.

Obviously $f^{-1}(y) \neq \phi$ and $f^{-1}(y) \in \mathring{\overline{C}}$. This is a contradiction.

Theorem 2.8—If $f : (X, \tau) \to (Y, \sigma)$ is a closed one-to-one "s. α. i." mapping then (X, τ) is an α-space.

PROOF : Let $L \in \tau_\alpha$, then $L = A - N$, where $A \in \tau$ and N is nowhere dense. By Lemma 2.7 $f(N)$ is nowhere dense and then by Theorem 2.3 $f^{-1}(f(N)) = N$ is closed. Therefore, $L = A - N = A \cap (X - N)$ is open.

The following example shows that in Theorem 2.8 one-to-one of f cannot be dropped.

Example 2.3—Let $X = \{a, b, c, d\}$; $\tau = \{\phi, X, \{a, b\}\}$; $Y = \{1, 2, 3\}$ and $\sigma = \{\phi, Y, \{2, 3\}\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = 2$; $f(b) = 3$; $f(c) = f(d) = 1$. Then f is a closed "s. α. i." mapping not one-to-one and (X, τ) is not an α-space.

Theorem 2.9—Let (Y, σ) be a regular α-space. The following statements are equivalent:

(i) $f : (X, \tau) \to (Y, \sigma)$ is "s. α. i.";

(ii) $f : (X, \tau) \to (Y, \sigma)$ is "α. i.";

(iii) $f : (X, \tau) \to (Y, \sigma)$ is "α. c.".

PROOF : (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) : Obvious.

(iii) $\Rightarrow$ (i) : Since (Y, σ) is a regular space, f is continuous (Mashhour *et al.*[6], Remark) and then $f : (X, \tau) \to (Y, \sigma)$ is continuous.

This completes the proof by Theorem 2.1.

Theorem 2.10—Let $f : (X, \tau) \to (Y, \tau)$ and $g : (Y, \sigma) \to (Z, \rho)$ be mappings.

(a) If f is "s. α. i." and g is "α. i.", then $g \circ f$ is "s. α. i.";

(b) If f is continuous and g is "s. α. i.", then $g \circ f$ is "s. α. i.":

(c) If f is "s. α. i." and g is "α. c.", then $g \circ f$ is continuous;

(d) If f is "α. c." and g is "s. α. i." then $g \circ f$ is "α. i."

PROOF : (a). $f : (X, \tau) \to (Y, \sigma_\alpha)$ and $g : (Y, \sigma_\alpha) \to (Z, \rho_\alpha)$ are continuous so that $g \circ f : (X, \tau) \to (Z, \rho_\alpha)$ is continuous.

(b) $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho_\alpha)$ are continuous so that $g \circ f (X, \tau) \to (Z, \rho_\alpha)$ is continuous.

(c) $f : (X, \tau) \to (Y, \sigma_\alpha)$ and $g: (Y, \sigma_\alpha) \to (Z, \rho)$ are continuous so that $g \bullet f : (X, \tau) \to (Z, \rho)$ is continuous.

(d) $f : (X, \tau_\alpha) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho_\alpha)$ are continuous so that $g \circ f : (X, \tau_\alpha) \to (Z, \rho_\alpha)$ is continuous.

Corollary—The composition of "s. α. i" mappings is "s. α. i.".

Remark 2.3 : If $f : (X, \tau) \to (Y, \sigma)$ is "s. α. i." and $A \subseteq X$, then :

(a) $f/A : (A, \tau/A) \to (Y, \sigma)$ is "s. α. i." because $(f/A)^{-1}(M) = f^{-1}(M) \cap A$, for each $M \subseteq Y$;

(b) $f : (X, \tau) \to (f(X), \sigma/f(X)$ is "s. α. i." because $(\sigma/B)_\alpha \subseteq (\sigma_\alpha) B$ for each $B \subseteq Y$ by (Prop. 2.1) of Faro[4].

If $f : (X, \tau) \to (Y, \sigma)$ is continuous mapping and $\sigma_1 \subseteq \sigma$, then $f : (X, \tau) \to (Y, \sigma_1)$ is continuous.

The following example shows that the above fact is not true in general for 's.α.i." mappings.

Example 2.4—Let $X = \{a, b, c, d\}$; $\tau = \sigma = \{\phi, X, \{a, b\}, \{c,d\}\}$ and $\sigma_1 = \{\phi, X, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity mapping, then f is "s. α. i" but $f : (X, \tau) \to (Y, \sigma_1)$ is not "s. α. i.".

Theorem 2.11—Let $f : (X, \tau) \to (\prod_{i \in I} X_i, \prod_{i \in I} \tau_i)$ be a "s. α. i." mapping.

Let $f_i : (X, \tau) \to (X_i, \tau_i)$ be a mapping defined as follows :

$f_i(x) = x_i$ if $f(x) = (x_i)_{i \in I}$, for each $i \in I$. Then f_i is "s. α. i." for each $i \in I$.

PROOF : Let P_i denote the projection of $\prod_{i \in I} X_i$ onto X_i. Then obviously $f_i = P_i \circ f$ for each $i \in I$. Since f is "s. α. i." and P_i is "α. i.", for each $i \in I$, f_i is "s. α. i" by Theorem 2.10.

The following example shows that converse to Theorem 2.11 is not true in general.

Example 2.5—Let $X_1 = X_2 = \{a, b\})$; $\tau_1 = \{\phi, X_1, \{a\}\}$ and $\tau_2 = \{\phi, X_2, \{b\}\}$. Define $f : (X_1, \tau_1) \to (X_1 \times X_2, \tau_1, \times \tau_2)$ by $f(a) = (a, a)$ and $f(b) = (b, a)$.

Then f is not "s. α. i." because $\{(a, b), (b, a)\} \in (\tau_1 \times \tau_2)_\alpha$ and $f^{-1}(\{(a, b), (b, a)\}) = \{b\} \notin \tau_1$. But $f_1 = P_1, \circ f : (X_1, \tau_1) \to (X_1, \tau_1)$ (the identity mapping) and $f_2 = P_2 \circ f: (X_1,\tau_1) \to (X_2, \tau_2)$ $(f_2(a) = a ; f_2(b) = a)$ are "s, α. i.".

Theorem 2.12—Let $f_i : (X_i \tau_i) \to (Y_i, \sigma_i)$ be a mapping for each $i \in I$.

Let $f: (\prod_{i\in I} X_i, \prod_{i\in I} \tau_i) \to (\prod_{i\in I} Y_i, \prod_{i\in I} \sigma_i)$ be a mapping defined as follows : $f(\{x_i\}) = \{f_i(x_i)\}$ for each $\{x_i\} \in \prod_{i\in I} X_i$. If f is "s. α. i.", then f_i is "s. α. i." for each $i \in I$.

PROOF : For each $V_i \in (\sigma_i)_\alpha$, we have $\prod_{j\neq i} Y_j \times V_i \in (\prod_{j\neq i} \sigma_i)_\alpha$ (see J Dugindji, Topology, p. 99,1.2 (2) and p. 105, Problems Section 1.2.),

Since f is "s. α. i.", $f^{-1}(\prod_{j\neq i} Y_j \times V_i) = \prod_{j\neq i} X_j \times f_i^{-1}(V_i)$ is open in $\prod_{i\in I} X_i$

and hence $f_i^{-1}(V_i)$ is open in X_i. This shows that f_1 is "s. α. i".

The following example shows that the converse to the Theorem 2.12 is not true in general.

Example 2.6—Let (X_1, τ_1) and (X_2, τ_2) be the topological spaces of example 2.5. Define $f_1 : (X_1, \tau_1) \to (X_1, \tau_1)$ by $f_1(a) = a$ and $f_1(b) = b$ and $f_2 : (X_2, \tau_2) \to (X_2, \tau_2)$ by $f_2(a) = f_2(b) = (a)$. Then f_1 and f_2 are "s. α. i." but $f: (X_1 \times X_2, \tau_1 \times \tau_3) \to (X_1 \times X_2, \tau_1 \times \tau_2)$ defined by $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ is not "s. α. i." because $\{(a, b), (b, a)\} \in (\tau_1 \times \tau_2)_\alpha$ and $f^{-1}\{(a, b), (b,a)\} = \{(b, a), (b,b)\} \notin (\tau_1 \times \tau_2)$.

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COMPLEMENTATION IN THE LATTICE OF ČECH CLOSURE OPERATORS

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In this paper we study some properties of the lattice of Čech closure operators on a fixed set, with special reference to complementation. It is proved that this lattice is complemented if and only if the set is finite. Further, it is shown that no element in this lattice has more than one complement.

INTRODUCTION

A Čech closure operator V on a set X is a function $V: P(X) \to P(X)$, where $P(X)$ is the power set of X, such that

(1) $V(\phi) = \phi$;

(2) $A \subset V(A)$ for all A in $P(X)$;

(3) $V(A \cup B) = V(A) \cup V(B)$ for all A, B in $P(X)$.

For brevity we call V a closure operator on X. Also (X, V) is called a closure space. A subset A of X is called open in (X, V) if $V(X \setminus A) = X \setminus A$. The set of all open sets in (X, V) is a topology on X, called the topology associated with V.

We denote the set of all closure operators on X by $L(X)$. It can be seen that $L(X)$ is a complete lattice under the partial order "$\leqslant$" on it defined by[3] $V_1 \leqslant V_2$ if and only if $V_2(A) \subset V_1(A)$ for every A in $P(X)$.

The concept of a closure operator is a natural generalization of that of a topology. The set of all topologies on a fixed set also forms a complete lattice under the natural order of set inclusion. The lattice of topologies has been investigated by several authors[2,4,6-9]. These investigations were mainly centred around the complementation problem in this lattice.

In this paper we conduct an analogous study of some properties of the lattice $L(X)$ with special reference to complementation. We determine the atoms and the dual atoms of this lattice. Then it is proved that the lattice $L(X)$ is complemented if and

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only if X is finite. It is then observed that $L(X)$ is always dually atomistic, but not atomistic when X is infinite. Finally we prove that no element in this lattice has more than one complement.

1. Infra and Ultra Closure Operators

The discrete closure operator D on X is defined by $D(A) = A$ for every A in $P(X)$. It is the largest element in $L(X)$. The indiscrete closure operator I on X is defined by $I(A) = \phi$ if $A = \phi$ and $I(A) = X$ otherwise. It is the smallest element in $L(X)$.

A closure operator on X, other than I, is called an infra closure operator if the only closure operator on X, strictly smaller than it, is I.

A closure operator on X, other then D, is called an ultra closure operator if the only closure operator, strictly larger than it, is D.

Note that the infra closure operators and the ultra closure operators are precisely the atoms and the dual atoms respectively of the lattice $L(X)$.

Notation

For a, b in X, $a \neq b$, let

$$\begin{aligned} V_{a,b}(A) &= \phi \text{ if } A = \phi \\ &= X \setminus \{b\} \text{ if } A = \{a\} \\ &= X \text{ otherwise.} \end{aligned}$$

It can be verified that $V_{a,b}$ is a closure operator on X.

Now we characterise an infra closure operators on X.

Theorem 1—A closure operator on X is an infra closure operator if and only if it is of the form $V_{a,b}$ for a, b in X, $a \neq b$.

Proof : If V is a closure operator on X, strictly smaller than $V_{a,b}$, then $V(\{a\})$ will be strictly larger than $X \setminus \{b\}$ and hence equal to X. Also $V(A) = X$ for every A in $P(X)$ other than ϕ and $\{a\}$. Hence $V = I$. Thus all closure operators of the form $V_{a,b}$ are infra closure operators.

Now let V be any closure operator on X other than I. Then there exists a non-empty subset A of X such that $V(A) \neq I(A) = X$. Now choose an element a in A and an element b in $X \setminus V(A)$. Then it can be verified that $V_{a,b} \leqslant V$. Thus all infra closure operators are of the form $V_{a,b}$ for some a, b in X, $a \neq b$.
Now let us proceed to characterise an ultra closure operator.

A topology T on X, which is not discrete, is called an ultratopology if the discrete topology is the only topology strictly larger than T. Fröhlich[4] proved that the ultratopologies on X are precisely the topologies of the form $P(X \setminus \{a\}) \cup \mathscr{U}$ where $a \in X$ and $\mathscr{U}$ is an ultrafilter on X which does not contain $\{a\}$.

To every topology T on X, we can associate a closure operator V on X defined by $V(A) = \bar{A}$ for all A in $P(X)$, where $\bar{A}$ is the closure of A in (X, T).

The closure operator associated with a topology on X is discrete if and only if it is the discrete topology.

The closure operator V associated with the ultratopology $P(X \setminus \{a\}) \cup \mathscr{U}$ is given by

$$V(A) = A \text{ if } a \in A \text{ or } X \setminus A \in \mathscr{U}$$
$$= A \cup \{a\} \text{ otherwise.}$$

Theorem 2—A closure operator on X is an ultra closure operator if and only if it is the closure operator associated with some ultratopology on X.

PROOF : Let $P(X \setminus \{a\}) \cup \mathscr{U}$ be an ultratopology on X and V the associated closure operator. Let V' be a closure operator on X strictly larger than V. Then there exists a subset A of X such that $V'(A) \subset V(A)$ but $V'(A) \neq V(A)$. Then $V(A) = A \cup \{a\}$ and $V'(A) = A$, which means $X \setminus A$ is open in (X, V') and not open in (X, V). Also every open set in (X, V) is open in (X, V'). Thus the associated topology of V' is strictly larger than the ultratopology and hence is discrete. But then $V' = D$. Hence the closure operators associated with ultratopologies are ultra closure operators.

Now let V be any closure operator on X other than D. Then there exists an element a in X such that $\{a\}$ is not open in (X, V). Now consider

$$\mathscr{F} = \{A \subset X : a \notin V(X \setminus A)\}.$$

It can be verified that $\mathscr{F}$ is a filter on X which contains neither $\{a\}$ nor $X \setminus \{a\}$. Then $\mathscr{F} \cup \{X \setminus \{a\}\}$ is a family with finite intersection property. For, otherwise there will be an F in $\mathscr{F}$ such that $F \cap (X \setminus \{a\}) = \phi$. Then $F \subset \{a\}$. Thus $\{a\} \in \mathscr{F}$. This is a contradiction. Using Zorn's lemma, we can prove that $\mathscr{F} \cup \{X \setminus \{a\}\}$ is contained in an ultrafilter $\mathscr{U}$ on X. Clearly $\mathscr{U}$ does not contain $\{a\}$. Now let V' be the closure operator associated with the ultratopology $P(X \setminus \{a\}) \cup \mathscr{U}$. Then $V \leqslant V'$. For, otherwise there exists a nonempty subset M of X such that $V'(M) \not\subset V(M)$. But then $a \in V'(M)$ and $a \notin V(M)$. Since $a \notin V(M)$, $X \setminus M \in \mathscr{F} \subset \mathscr{U}$. Then $V'(M) = M$, a contradiction which proves the theorem.

Remark : Note that in the course of proofs of Theorems 1 and 2 we also proved that every element of the lattice $L(X)$, other than I and D, is larger than or equal to an atom and smaller than or equal to a dual atom.

Definition—Let $x \in X$. Then the set

$$\mathscr{U}(x) = \{A \subset X : x \in A\}$$

is an ultrafilter on X. Such ultrafilters are called principal ultrafilters. An ultratopology $P(X \setminus \{a\}) \cup \mathscr{U}$ on X is called a principal or nonprincipal ultratopology accord-

ing as $\mathcal{U}$ is principal or not. The closure operator associated with an ultratopology is called principal or nonprincipal ultra closure operator according as the ultratopology is principal or not.

Theorem 3—Infra closure operators are smaller than or equal to any nonprincipal ultra closure operator.

Proof : Let $V_{x,y}$ be an infra closure operator and V a non-principal ultra closure operator. Since $V_{x,y}\ (A) = X$ for all A in $P(X)$ other than ϕ and $\{x\}$, we need only show that

$$V(\{x\}) \subset V_{x,y}(\{x\}) = X \setminus \{y\}.$$

But since all nonprincipal ultratopologies are[9] T_1, $V\ (\{x\}) = \{x\}$ for all x in X Hence the result.

Definition—A closure operator V on a set X is called a T_1 closure operator if $V\ (\{x\}) = \{x\}$ for every x in X.

Remark : A closure operator is T_1 if and only if the associated topology is T_1. It can be seen that a closure operator is T_1 if and only if it is larger than every infra closure operator. The proof of the necessity is similar to the proof of Theorem 3 and the sufficiency part is easy.

Notation : We denote the principal ultra closure operator associated with the principal ultratopology $P(X \setminus \{a\}) \cup \mathcal{U}(b)$, $a \neq b$, by $T_{a,b}$.

Theorem 4—An infra closura operator $V_{x,y}$ is smaller than or equal to a principal ultra closure operator $T_{a,b}$ if and only if $x \neq b$ or $y \neq a$.

Proof : We have $V_{a,b}(\{a\}) = X \setminus \{b\}$. But $T_{b,a}(\{a\}) = \{a, b\}$. Then $T_{b,a}(\{a\}) \not\subset V_{a,b}(\{a\})$. Thus $V_{a,b} \not\leqslant T_{b,a}$.

Now when A is a nonempty subset of X other than $\{x\}$, $T_{a,b}\ (A) \subset X = V_{x,y}\ (A)$. Also when $x \neq b$, $T_{a,b}(\{x\}) = \{x\} \subset X \setminus \{y\} = V_{x,y}\ (\{x\})$ and when $y \neq a$, $T_{a,b}(\{x\}) \subset \{x, a\} \subset X \setminus \{y\} = V_{x,y}(\{x\})$. Thus $V_{x,y} \leqslant T_{a,b}$ if $x \neq b$ or $y \neq a$.

2. Complementation in the Lattice $L(X)$

Hartmanis[6] proved that the lattice of topologies on a finite set is complemented and raised the question about the complementation in the lattice of topologies on an arbitrary set. This problem was solved affirmatively by Steiner[8]. Van Rooij[10] gave a simpler proof independently. In this section we study the complementation in the lattice $L(X)$.

Theorem 5—$L(X)$ is complemented if and only if X is finite.

Proof : Let X be finite. To every closure operator V on X, we can associate a reflexive relation $R(V)$ on X such that $a\ R(V)\ b$ if and only if $b \in V(\{a\})$. Then R is

a dual isomorphism from the lattice $L(X)$ onto the lattice of all reflexive relations on X under the partial order of set inclusion[3]. But the lattice of all reflexive relation on X is isomorphic to the lattice of all subsets of the set $\{(x, y): x \in X, y \in Y, x \neq y\}$, the isomorphism being the function which maps each reflexive relation R on X to $R \setminus \{(x, x): x \in x\}$. Since the latter lattice is complemented, the lattice of reflexive relations and consequently the lattice $L(X)$ is complemented.

Now let X be an infinite set. Then in view of Theorem 2, we can prove that there exists a nonprincipal ultra closure operator on X. By Theorem 3 every nonprincipal ultra closure operator is larger than each atom of $L(X)$. Thus $L(X)$ is not complemented.

Remark : In the proof of the sufficiency part of the previous theorem we see that $L(X)$ is dually isomorphic to the lattice of all subsets of a set under the order of set inclusion. Thus $L(X)$ is a Boolean lattice when X is finite.

But the lattice of topologies on X is not even modular when X contains atleast three elements. Also it is not self dual when X contains atleast four elements[9].

Eventhough the lattice $L(X)$ is not complemented, some elements do have complements. For example, if x and y are in X, $x \neq y$, $V_{x,y}$ and $T_{y,x}$ are complements of each other by Theorem 4.

Definitions—A lattice L with the largest element 1 and the smallest element 0 is called atomistic if every element of L other than 0 is the supremum of the set of all atoms of L less than or equal to it. It is dually atomistic if every element of L other than 1 is the infimum of the set of all dual atoms of L larger than it.

Remark : The lattice of topologies on X is both atomistic and dually atomistic[4]. The lattice $L(X)$ is atomistic if and only if X is finite, by the previous remark and Theorem 3.

Theorem 6—The lattice $L(X)$ is dually atomistic.

PROOF : Let V be an element of $L(X)$ other than D. Let V' be the infimum of the set of all ultra closure operators larger than or equal to V. Clearly $V \leqslant V'$. It suffices to prove that $V = V'$. Suppose not. Then there exists a nonempty subset A of X and an element a of $V(A)$ such that $a \notin V'(A)$. Now let

$$\mathcal{F} = \{F \subset X : a \notin V(X \setminus F)\}.$$

It can be verified that $\mathcal{F}$ is a filter on X such that $A \notin \mathcal{F}$ and $X \setminus A \notin \mathcal{F}$.

Now $\mathcal{F} \cup \{A\}$ is a family with finite intersection property, for otherwise there exists $F \in \mathcal{F}$ such that $F \cap A = \phi$ and then $X \setminus A \in \mathcal{F}$, as $F \subset X \setminus A$, which is a contradiction. Then by the use of Zorn's lemma, we can prove that $\mathcal{F} \cup \{A\}$ is

contained in an ultrafilter $\mathscr{U}$ on X. Clearly $\mathscr{U}$ does not contain $\{a\}$ for $A \in \mathscr{U}$ and $a \notin A$.

Let S be the ultra closure operator associated with the ultratopology $P\ (X \setminus \{a\}) \cup \mathscr{U}$. Then $S\ (M) \subset M \cup \{a\}$ for every M in $P\ (X)$. Also, when $a \notin V\ (M)$, $X \setminus M \in \mathscr{F} \subset \mathscr{U}$ and hence $S\ (M) = M$. Thus $S\ (M) \subset V\ (M)$ for every M in $P\ (X)$. That is $V \leqslant S$. But $V' \notin S$ for $S\ (A) = A \cup \{a\}$ as $X \setminus A \notin \mathscr{U}$ and $a \notin A$ where $a \notin V'\ (A)$. This contradicts the definition of V'. Hence the result.

Theorem 7—If V and V' are elements of $L\ (X)$ such that $V \wedge V' = I$, then every ultra closure operator is larger than or equal to one of V and V'.

PROOF : On the contrary assume that there exists an ultra closure operator S associated with an ultratopology $P\ (X \setminus \{a\}) \cup \mathscr{U}$ such that $S \ngeqslant V$ and $S \ngeqslant V'$. Then $S\ (A) \not\subset V\ (A)$ for some nonempty subset A of X since $S \ngeqslant V$. But $S\ (A) \subset A \cup \{a\}$. Therefore $a \notin V\ (A)$. Also $A \in \mathscr{U}$, for otherwise $X \setminus A \in \mathscr{U}$ and hence $S\ (A) = A \subset V\ (A)$, a contradiction. Similarly there exists a nonempty subset B of X such that $a \notin V\ (B)$ and $B \in \mathscr{U}$. Since both A and B are elements of the filter $\mathscr{U}$, $A \cap B \neq \phi$. Choose $x \in A \cap B$. Then, $V_{x,a} \leqslant V$ and $V_{x,a} \leqslant V'$, which is a contradiction since $V \wedge V' = I$. Hence the result.

Theorem 8—In the lattice $L\ (X)$, no element has more than one complement.

PROOF : Let a closure operator V on X, other than I and D, have complements V_1 and V_2. Then both the set of all ultra closure operators larger than or equal to V_1 and the set of all ultra closure operators larger than or equal to V_2 are the same as the set of all ultra closure operators which are neither larger than nor equal to V, by Theorem 7. But then $V_1 = V_2$ by Theorem 6. Hence the result.

Remark : In the lattice of topologies on a fixed set X, the complements are not unique in general[7].

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CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS II

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Let T denote the class of functions

$$f(z) = a_1 z - \sum_{m=2}^{\infty} a_m z^m, \; a_m \geqslant 0, \; a_1 > 0$$

analytic in the unit disc E. Let T_1, T_2 denote subclasses of T satisfying $f(z_0) = z_0$ and $f'(z_0) = 1$ $(-1 < z_0 < 1)$ respectively. Properties of certain subclasses of T_1 and T_2 are investigated and sharp results are obtained.

INTRODUCTION

Let S denote the class of functions $f(z)$ analytic in the unit disc $E = \{z : |z| < 1\}$ with $f(0) = 0$. The Hadamard product $(f * g)(z)$ of two functions $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$ in S is given by $(f * g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m$.

Let $D^\alpha f(z) = \dfrac{z}{(1-z)^{\alpha+1}} * f(z)$, $(\alpha > -1)$. Ruscheweyh[5] observed that $D^n f(z) = z(z^{n-1} f(z))^{(n)}/n!$ when $n \in N \cup \{0\}$, where $N = \{1, 2, 3, \ldots\}$. This symbol $D^n f(z)$, $(n \in N \cup \{0\})$ was called the nth order Ruscheweyh derivative of $f(z)$ by Al-Amiri[1]. We observe that $D^0 f = f$ and $D^1 f = zf'$.

Let T be the subclass of S consisting functions of the form $f(z) = a_1 z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$, $a_1 > 0$. Let $S_n(A, B)$ denote the class of functions $f \in T$ such that

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \; -1 \leqslant A < B \leqslant 1,$$

where $n \in N \cup \{0\}$ and $w \in H = \{w \text{ analytic}, w(0) = 0 \text{ and } |w(z)| < 1, z \in E\}$. Let $K_n(A, B)$ denote the class of functions $f \in T$ such that $zf'(z) \in S_n(A, B)$.

For a given real number z_0 $(-1 < z_0 < 1)$ let T_1 and T_2 be the subclasses of T satisfying $f(z_0) = z_0$ and $f'(z_0) = 1$, respectively. Let $S_1(z_0)$, $S_2(z_0)$, $K_1(z_0)$ and $K_2(z_0)$ be the subclasses of T defined as follows :

$$S_i(z_0) = S_n(A, B) \cap T_i, \; K_i(z_0) = K_n(A, B) \cap T_i, \; i = 1, 2.$$

In this paper we obtain necessary and sufficient conditions for functions to be in $S_n(A, B)$, $K_n(A, B)$, $S_i(z_0)$ and $K_i(z_0)$, $i = 1, 2$. We determine radius of convexity for the classes $S_i(z_0)$, $i = 1, 2$. Also closure theorems are proved for these subclasses. Further we determine a necessary and sufficient condition that a subset X of the real interval $(0, 1)$ should satisfy the property that $\bigcup\limits_{z_i \in X} S_1(z_i)$, $\bigcup\limits_{z_i \in X} S_2(z_i) \cup \bigcup\limits_{z_i \in X} K_1(z_i)$ and $\bigcup\limits_{z_i \in X} K_2(z_i)$ each forms a convex family. For these classes extreme points are also determined.

The results obtained by Silverman[7] can be deduced from the corresponding results in this paper by taking $B = 1$, $A = 2\gamma - 1$ $(0 \leqslant \gamma < 1)$ and $n = 0$. The classes studied by Gupta and Jain[2], Owa[4], Silverman[6] and Gupta and Jain[3] can be obtained from $S_i(z_0)$ and $K_i(z_0)$ by taking proper choices for A, B, n and z_0.

2. The Main Theorems

We now introduce the following notations for brevity :

$$D_m = (n + m - 1)!\,[(B + 1)(n + m) - (A + 1)(n + 1)]$$

$$E_m = (n + 1)!\,(m - 1)!\,(B - A), \quad F_m = D_m - E_m z_0^{m-1}.$$

Theorem 2.1—A function $f(z) \in T$ is in $S_n(A, B)$ if and only if

$$\sum_{m=2}^{\infty} \frac{D_m a_m}{E_m} \leqslant a_1. \qquad \ldots(2.1)$$

Proof : Suppose $f \in S_n(A, B)$. Then

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad -1 \leqslant A < B \leqslant 1,\ w(z) \in H,\ z \in E.$$

From this we get

$$w(z) = \frac{D^n f(z) - D^{n+1} f(z)}{BD^{n+1} f(z) - AD^n f(z)}$$

and $|w(z)| < 1$ implies

$$|w(z)| = \left| \frac{\sum\limits_{m=2}^{\infty} \dfrac{(n + m - 1)!}{(n + 1)!\,(m - 1)!} [(n + m) - (n - 1)]\, a_m z^m}{(B - A)\, a_1 z - \sum\limits_{m=2}^{\infty} \dfrac{(n + m - 1)!}{(n + 1)!\,(m - 1)!} [B(n + m) - A(n + 1)]\, a_m z^m} \right|$$

$$< 1. \qquad \ldots(2.2)$$

Therefore

$$\operatorname{Re}\left\{\frac{\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!\,(m-1)!}[(n+m)-(n+1)]\,a_m z^{m-1}}{(B-A)\,a_1-\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!\,(m-1)!}[B(n+m)-A(n+1)]a_m z^{m-1}}\right\} < 1. \qquad \ldots(2.3)$$

We consider real values of z and take $z = r$ with $0 \leqslant r < 1$. Then, for $r = 0$, denominator of (2.3) is positive and so it is positive for all r with $0 \leqslant r < 1$, since $w(z)$ is analytic for $|z| < 1$. Then (2.3) gives

$$\sum_{m=2}^{\infty}\frac{D_m\,a_m\,r^{m-1}}{E_m} < a_1. \qquad \ldots(2.4)$$

Letting $r \to 1$, we get (2.1).

Conversely, suppose $f \in T$ and f satisfies (2.1). For $|z| = r$, $0 \leqslant r < 1$, (2.4) is implied by (2.1), since $r^{m-1} < 1$. So we have

$$\left|\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!\,(m-1)!}[(n+m)-(n+1)]\,a_m z^{m-1}\right|$$

$$\leqslant \sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!\,(m-1)!}[(n+m)-(n+1)]\,a_m r^{m-1}$$

$$< (B-A)\,a_1 - \sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!\,(m-1)!}[B(n+m)$$

$$-A(n+1)]\,a_m r^{m-1}$$

$$\leqslant \left|(B-A)\,a_1 - \sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!\,(m-1)!}[B(n+m)\right.$$

$$\left.-A(n+1)]\,a_m z^{m-1}\right|$$

which gives (2.2) and hence follows that

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1+Aw(z)}{1+Bw(z)},\ w \in H,\ z \in E,\ -1 \leqslant A < B \leqslant 1.$$

That is, $f \in S_n(A, B)$.

Corollary 2.1—A function $f(z) \in T$ is in $K_n(A, B)$ if and only if

$$\sum_{m=2}^{\infty} \frac{mD_m a_m}{E_m} \leqslant a_1. \qquad \text{...(2.5)}$$

Theorem 2.2—A function $f(z) \in T_1$ is in $S_1(z_0)$ if and only if

$$\sum_{m=2}^{\infty} \frac{F_m a_m}{E_m} \leqslant 1. \qquad \text{...(2.6)}$$

PROOF: Let $f(z) \in S_1(z_0)$. Then for fixed z_0 $(-1 < z_0 < 1)$, $f(z_0) = a_1 z_0 - \sum_{m=2}^{\infty} a_m z_0^m$. Since $f(z_0) = z_0$, we have $a_1 = 1 + \sum_{m=2}^{\infty} a_m z_0^{m-1}$. Since $f \in S_1(z_0)$, $f \in S_n(A, B)$ and so from Theorem 2.1., using the relation $a_1 = 1 + \sum_{m=2}^{\infty} a_m z_0^{m-1}$, we get (2.6).

Conversely, let $f \in T_1$ and let (2.6) be satisfied. Since $f(z_0) = z_0$, we get $\sum_{m=2}^{\infty} a_m z_0^{m-1} = a_1 - 1$. Substituting for $(a_1 - 1)$ in (2.6) we get (2.1). By Theorem 2.1., we have $f \in S_n(A, B)$ and hence $f \in S_n(A, B) \cap T_1 = S_1(z_0)$.

Corollary 2.2—If $f \in S_1(z_0)$, then

$$a_m \leqslant \frac{E_m}{F_m} \quad (m = 2, 3, \ldots),$$

with equality for $f(z) = \dfrac{D_m z - E_m z^m}{F_m}$.

Theorem 2.3—Let $f \in T_1$. Then $f \in K_1(z_0)$ if and only if

$$\sum_{m=2}^{\infty} \frac{(mD_m - E_m z_0^{m-1})}{E_m} a_m \leqslant 1.$$

PROOF: This follows from Corollary 2.1 as Theorem 2.2 follows from Theorem 2.1.

Theorem 2.4—Let $f \in T_2$. Then $f \in S_2(z_0)$ if and only if

$$\sum_{m=2}^{\infty} \frac{\left(D_m - mE_m z_0^{m-1}\right) a_m}{E_m} \leqslant 1. \qquad \text{...(2.7)}$$

PROOF : Suppose $f \in S_2(z_0)$. Then for fixed z_0 $(-1 < z_0 < 1)$, $f'(z_0) = a_1 - \sum_{m=2}^{\infty} m a_m z_0^{m-1}$. Since $f'(z_0) = 1$, we have $a_1 = 1 + \sum_{m=2}^{\infty} m a_m z_0^{m-1}$. $f \in S_2(z_0)$ implies $f \in S_n(A, B)$ and so Theorem (2.1) holds for f. Hence substituting $a_1 = 1 + \sum_{m=2}^{\infty} m a_m z_0^{m-1}$ in (2.1) we get (2.7).

Conversely, let $f \in T_2$ and f satisfy (2.7). Since $f'(z_0) = 1$, we have $\sum_{m=1}^{\infty} m a_m z_0^{m-1} = a_1 - 1$. Substituting the value of $\sum_{m=2}^{\infty} m a_m z_0^{m-1}$ in (2.7), we get (2.1). From Theorem 2.1, $f \in S_n(A, B)$. Hence $f \in S_2(z_0)$.

Theorem 2.5—Let $f \in T_2$. Then $f \in K_2(z_0)$ if and only if

$$\sum_{m=2}^{\infty} \frac{m F_m a_m}{E_m} \leqslant 1.$$

PROOF : This follows from Corollary 2.1 as Theorem 2.4 follows from Theorem 2.1.

3. Closure Theorems

Theorem 3.1—The class $S_1(z_0)$ is closed under convex linear combination.

PROOF : Let the functions $f_i(z) = a_{1,i} z - \sum_{m=2}^{\infty} a_{m,i} z^m$ $(a_{m,i} \geqslant 0, a_{1,i} > 0)$ be in the class $S_1(z_0)$ for $i = 1, 2, \ldots k$. We have to show that if the function h is defined by $h(z) = \sum_{i=1}^{k} b_i f_i(z)$ $(b_i \geqslant 0)$ where $\sum_{i=1}^{k} b_i = 1$, then h also belongs to the class $S_1(z_0)$. From the definition of $h(z)$ we have

$$h(z) = d_1 z - \sum_{m=2}^{\infty} d_m z^m$$

where $d_m = \sum_{m=2}^{\infty} b_i a_{m,i}$, $m = 1, 2, \ldots$. Since $f_i(z)$ are in $S_1(z_0)$, $i = 1, 2, \ldots, k$, we have from Theorem 2.2,

$$\sum_{m=2}^{\infty} \frac{F_m a_{m,i}}{E_m} \leqslant 1, i = 1, 2, \ldots k.$$

Therefore we have

$$\sum_{m=2}^{\infty} \frac{F_m}{E_m}\left(\sum_{l=1}^{k} b_l \, a_{m,l}\right) = \sum_{l=1}^{k} b_l \left(\sum_{m=2}^{\infty} \frac{F_m \, a_{m,l}}{E_m}\right)$$

$$< \sum_{l=1}^{k} b_l = 1.$$

This shows that the function h belongs to the class $S_1(z_0)$ and the theorem is proved.

We can prove in the same way the following:

Theorem 3.2—The classes $S_2(z_0)$, $K_1(z_0)$ and $K_2(z_0)$ are closed under convex linear combinations.

Theorem 3.3—Define

$$f_1(z) = z \text{ and } f_m(z) = \frac{D_m z - E_m z^m}{F_m}, \; (m = 2, 3, \ldots).$$

Then $f(z) \in S_1(z_0)$ if and only if it can be expressed in the form $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$, where each $\mu_m \geqslant 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

PROOF: Suppose $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$, where $\mu_m \geqslant 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$. Then

$$f(z) = \mu_1 f_1(z) + \sum_{m=2}^{\infty} \mu_m f_m(z) = \mu_1 z + \sum_{m=2}^{\infty} \mu_m \frac{D_m z - E_m z^m}{F_m}$$

$$= \left(\mu_1 + \sum_{m=2}^{\infty} \frac{D_m}{F_m} \mu_m\right) z - \sum_{m=2}^{\infty} \frac{E_m \mu_m z^m}{F_m}.$$

Now

$$\sum_{m=2}^{\infty} \frac{F_m}{E_m} a_m = \sum_{m=2}^{\infty} E. \frac{E_m}{F_m} \mu_m = \sum_{m=2}^{\infty} \mu_m$$

$$= 1 - \mu_1 \leqslant 1.$$

Also by definition we have $f_m(z_0) = z_0$. Therefore

$$f(z_0) = \sum_{m=1}^{\infty} \mu_m f_m(z_0) = \sum_{m=1}^{\infty} \mu_m z_0 = z_0 \sum_{m=1}^{\infty} \mu_m = z_0.$$

This implies $f \in T_1$. By Theorem 2.2, $f \in S_1(z_0)$.

Conversely, let $f \in S_1(z_0)$. Then $a_1 = 1 + \sum_{m=2}^{\infty} a_m z_0^{m-1}$.

Define

$$\mu_m = \frac{F_m}{E_m} a_m, \; m \geqslant 2 \text{ and } \mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m.$$

From Theorem 2.2 we have $\sum_{m=2}^{\infty} \mu_m \leqslant 1$ and so $\mu_1 \geqslant 0$.

Now

$$f(z) = a_1 z - \sum_{m=2}^{\infty} a_m z^m = \mu_1 z + \sum_{m=2}^{\infty} \mu_m z \left[1 + \frac{(z_0^{m-1} - z^{m-1})}{\mu_m} a_m\right]$$

$$= \mu_1 z + \sum_{m=2}^{\infty} \mu_m \frac{D_m z - E_m z^m}{F_m}$$

$$= \mu_1 f_1(z) + \sum_{m=2}^{\infty} \mu_m f_m(z) = \sum_{m=1}^{\infty} \mu_m f_m(z).$$

This completes the proof of the theorem.

In a similar manner we can prove the following Theorems.

Theorem 3.4—Define

$$f_1(z) = z \text{ and } f_m(z) = \frac{D_m z - E_m z^m}{D_m - mE_m z_0^{m-1}}, \; (m = 2, 3, \ldots).$$

Then $f \in S_2(z_0)$ if and only if it can be expressed in the form $\sum_{m=1}^{\infty} \mu_m f_m(z)$, $z \in E$ where each $\mu_m > 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

Theorem 3.5—Define

$$f_1(z) = z \text{ and } f_m(z) = \frac{mD_m z - E_m z^m}{mD_m - E_m z_0^{m-1}}, \; (m = 2, 3, \ldots).$$

Then $f \in K_1(z_0)$ if and only if it can be expressed in the form $\sum_{m=1}^{\infty} \mu_m f_m(z)$, $z \in E$ where $\mu_m \geqslant 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

Theorem 3.6—Define

$$f_1(z) = z \text{ and } f_m(z) = \frac{D_m z - E_m z^m/m}{F_m}, \; (m - 2, 3, \ldots).$$

Then $f \in K_2(z_0)$ if and only if it can be expressed in the form $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$, $z \in E$, where each $\mu_m \geqslant 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

4. The Radius of Convexity of the Classes $S_i(z_0)$, $i = 1, 2$

Theorem 4.1—Let $f \in T$. If $f \in S_1(z_0)$ or $S_2(z_0)$, then f is convex in the disc $|z| < r$ where

$$r = \inf_m \left[\frac{D_m}{m^2 E_m} \right]^{1/(m-1)}.$$

The bound is sharp for the function $f_m(z) = \dfrac{D_m z - E_m z^m}{F_m}$.

Proof : To prove the theorem it is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} \right| \leqslant 1 \text{ for } |z| < r, \, z \in E.$$

We have

$$\left| \frac{zf''(z)}{f'(z)} \right| = \left| \frac{-\sum_{m=2}^{\infty} m(m-1) a_m z^{m-1}}{a_1 - \sum_{m=2}^{\infty} m a_m z^{m-1}} \right|$$

$$\leqslant \frac{\sum_{m=2}^{\infty} m(m-1) a_m |z|^{m-1}}{|a_1 - \sum_{m=2}^{\infty} m a_m z^{m-1}|}. \qquad \ldots(4.1)$$

Consider the values of z for which

$$|z| \leqslant \inf_m \left[\frac{D_m}{m^2 E_m} \right]^{1/(m-1)}$$

that is,

$$|z|^{m-1} < \frac{D_m}{m^2 E_m}$$

holds. Then

$$\sum_{m=2}^{\infty} m a_m |z|^{m-1} \leqslant \sum_{m=2}^{\infty} \frac{D_m}{m E_m} a_m.$$

Now

$$\sum_{m=2}^{\infty} m a_m |z|^{m-1} < a_1$$

provided

$$\sum_{m=2}^{\infty} \frac{D_m a_m}{m E_m} < a_1. \qquad \ldots(4.2)$$

Now if

$$f \in S_1(z_0), \sum_{m=2}^{\infty} \frac{F_m a_m}{E_m} \leqslant 1$$

or

$$\sum_{m=2}^{\infty} \frac{D_m}{E_m} a_m < 1 + \sum_{m=2}^{\infty} a_m z_0^{m-1} = a_1$$

and since

$$\sum_{m=2}^{\infty} \frac{D_m a_m}{m E_m} < \sum_{m=2}^{\infty} \frac{D_m}{E_m} a_m, \text{ (4.2) holds.}$$

Let $f \in S_2(z_0)$.

Then $\displaystyle\sum_{m=2}^{\infty} \frac{\left(D_m - m E_m z_0^{m-1}\right) a_m}{E_m} \leqslant 1$, by Theorem 2.4. Hence

$\displaystyle\sum_{m=2}^{\infty} \frac{D_m a_m}{E_m} \leqslant 1 + \sum_{m=2}^{\infty} m a_m z_0^{m-1} = a_1$ and so (4.2) holds if $f \in S_2(z_0)$. Therefore we can rewrite the denominator of the right hand side of inequality (4.1) for the considered values of z, using the fact that

$$a_1 > \sum_{m=2}^{\infty} m a_m |z|^{m-1}.$$

Thus

$$\left|\frac{z f''(z)}{f'(z)}\right| \leqslant \frac{\sum_{m=2}^{\infty} m(m-1) a_m |z|^{m-1}}{a_1 - \sum_{m=2}^{\infty} m a_m |z|^{m-1}} \leqslant 1 \text{ if}$$

$$\sum_{m=2}^{\infty} m^2 a_m |z|^{m-1} \leqslant a_1. \qquad \ldots(4.3)$$

If $f \in S_1 \ (z_0)$, (4.3) is equivalent to

$$\sum_{m=2}^{\infty}\left(m^2 \ |z|^{m-1} - z_0^{m-1} \right) a_m \leqslant 1. \qquad \ldots(4.4)$$

Again if $f \in S_2\,(z_0)$, (4.3) is equivalent to

$$\sum_{m=2}^{\infty}\left(m^2 \ |z|^{m-1} - m \ z_0^{m-1} \right) a_m \leqslant 1. \qquad \ldots(4.5)$$

By Theorem 2.2, $f \in S_1\,(z_0)$ if and only if

$$\sum_{m=2}^{\infty} \left(\frac{D_m}{E_m} - z_0^{m-1} \right) a_m \leqslant 1.$$

Hence inequality (4.4) is true if

$$m^2 \ |z|^{m-1} - z_0^{m-1} \leqslant \frac{D_m}{E_m} - z_0^{m-1}$$

for all m, that is, if

$$|z| \leqslant \left[\frac{D_m}{m^2 \ E_m} \right]^{1/(m-1)}$$

for all m. Again by Theorem 2.4, $f \in S_2\,(z_0)$ if and only if

$$\sum_{m=2}^{\infty} \left(\frac{D_m}{E_m} - m z_0^{m-1}\right) a_m \leqslant 1.$$

Again inequality (4.5) is true if

$$m^2 \ |z|^{m-1} - m z_0^{m-1} \leqslant \frac{D_m}{E_m} - m z_0^{m-1}$$

for all m, that is, if

$$|z| \leqslant \left[\frac{D_m}{m^2 E_m}\right]^{1/(m-1)}$$

for all m. This result is sharp for the extremal function

$$f_m(z) = \frac{D_m z - E_m z^m}{S_m}, \quad (m = 2, 3, \ldots).$$

Remark : The conclusion of Theorem 4.1 is independent of the point z_0.

5. Convex Families

Let X be a nonempty subset of the real interval $(0,1)$. We define $S_1(X)$ by

$$S_1(X) = \bigcup_{z_i \in X} S_1(z_i).$$

If X has only one element, then $S_1(X)$ is known to be a convex family by Theorem (3.1). We investigate, in the following, this class for other subsets X.

Lemma —If $f \in S_1(z_0) \cap S_1(z_1)$, where z_0 and z_1 are distinct positive numbers, then $f(z) = z$.

Proof : Let $f \in S_1(z_0) \cap S_1(z_1)$ and let $f(z) = a_1 z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$. Then

$$a_1 = 1 + \sum_{m=2}^{\infty} a_m z_0^{m-1} = 1 + \sum_{m=2}^{\infty} a_m z_1^{m-1}.$$

Since $a_m \geqslant 0$, $z_0 > 0$ and $z_1 \geqslant 0$, this implies $a_m = 0$ for $m > 2$. Hence the lemma.

Remark : The fact that if $f(z) \in S_1(z_0)$ and $f(z)$ is odd, then $f(z) \in S_1(-z_0)$ shows that the conclusion of the lemma need not follow if we relax the condition that the fixed points be positive.

Theorem 5.1—If X is contained in the interval $(0,1)$, then $S_1(X)$ is a convex family if and only if X is connected.

Proof : Let X be connected. Suppose $z_0, z_1 \in X$ with $z_0 \leqslant z_1$. To prove $S_1(X)$ is a convex family it suffices to show, for $f(z) = a_1 z - \sum_{m=2}^{\infty} a_m z^m \in S_1(z_0)$, $g(z) = b_1 z - \sum_{m=2}^{\infty} b_m z^m \in S_1(z_1)$ and $0 \leqslant \eta \leqslant 1$, that there exists a z_2 $(z_0 \leqslant z_2 \leqslant z_1)$ such that $h(z) = \eta f(z) + (1-\eta) g(z)$ is in $S_1(z_2)$. Since $f \in S_1(z_0)$ and $g \in S_1(z_1)$, we have $a_1 = 1 + \sum_{m=2}^{\infty} a_m z_0^{m-1}$ and $b_1 = 1 + \sum_{m=2}^{\infty} b_m z_1^{m-1}$ Therefore we have

$$t(z) = \frac{h(z)}{z} = 1 + \eta \sum_{m=2}^{\infty} a_m \left(z_0^{m-1} - z^{m-1} \right)$$

(equation continued on p. 170)

$$+ (1-\eta) \sum_{m=2}^{\infty} \left(z_1^{m-1} - z^{m-1} \right) \quad ...(5.1)$$

$t(z)$ is continuous function of z and is real when z is real with $t(z_0) \geqslant 1$ and $t(z_1) \leqslant 1$. Hence $t(z_2) = z_2$ for some z_2, $z_0 \leqslant z_2 \leqslant z_1$. This implies $h(z_2) = z_2$ for some z_2, $z_0 \leqslant z_2 \leqslant z_1$, that is $h(z) \in T_1$. Now, from (5.1) and $h(z_2) = z_2$, we have

$$\sum_{m=2}^{\infty} \left(\frac{D_m}{E_m} - z_2^{m-1} \right) [\eta a_m + (1-\eta) b_m]$$

$$= \eta \sum_{m=2}^{\infty} \left(\frac{D_m}{E_m} - z_0^{m-1} \right) a_m + (1-\eta) \sum_{m=2}^{\infty} \left(\frac{D_m}{E_m} - z_1^{m-1} \right) b_m$$

$$+ \eta \sum_{m=2}^{\infty} \left(z_0^{m-1} - z_2^{m-1} \right) a_m + (1-\eta) \sum_{m=2}^{\infty}$$

$$\times \left(z_1^{m-1} - z_2^{m-1} \right) b_m$$

$$= \eta \sum_{m=2}^{\infty} \left(\frac{D_m}{E_m} - z_0^{m-1} \right) a_m + (1-\eta) \sum_{m=2}^{\infty} \left(\frac{D_m}{E_m} - z_1^{m-1} \right) b_m$$

$$\leqslant \eta + (1-\eta) = 1$$

by Theorem 2.2, since $f \in S_1(z_0)$ and $g \in S_1(z_1)$. Hence we have $h \in S_1(z_2)$, by theorem 2.2. Since z_0, z_1 and η are arbitrary, the family $S_1(x)$ is convex.

Conversely, if X is not connected, then there exist z_0, z_1 and z_2 such that z_0, $z_1 \in X$, $z_2 \notin X$ and $z_0 < z_2 < z_1$. Assume $f \in S_1(z_0)$ and $g \in S_1(z_1)$ are not both the identity function. Then, for fixed z_2 and $0 \leqslant \eta \leqslant 1$, we have from (5.1).

$$t(\eta) = t(z_2, \eta) = 1 + \eta \sum_{m=2}^{\infty} a_m \left(z_0^{m-1} - z_2^{m-1} \right)$$

$$+ (1-\eta) \sum_{m=2}^{\infty} b_m \left(z_1^{m-1} - x_2^{m-1} \right).$$

Since $t(z_2, 0) > 1$ and $t(z_2, 1) < 1$, there exists a η_0, $0 < \eta_0 < 1$ such that $t(z_2, \eta_0) = 1$ or $h(z_2) = z_2$ where $h(z) = \eta_0 f(z) + (1-\eta_0) g(z)$. Thus $h \in S_1(z_2)$. From the lemma we have $h(z) \notin S_1(X)$, since $z_2 \notin X$ and $h(z) \neq z$. This implies the family $S_1(X)$ is not convex which is a contradiction.

Theorem 5.2—Let $[z_0, z_1] \subset (0,1)$. Then the extreme points of $S_1([z_0, z_1])$ are z,

$$f_m(z) = \frac{D_m z - E_m z^m}{D_m - E_m z_0^{m-1}}, \; (m = 2, 3, \ldots),$$

and

$$g_m(z) = \frac{D_m z - E_m z^m}{D_m - E_m z_1^{m-1}}, \ (m = 2, 3, \ldots).$$

PROOF : Since $S_1([z_0, z_1])$ is convex, a function $h \in S_1(z_2)$, $z_0 \leqslant z_2 \leqslant z_1$, can only be an extreme point of $S_1([z_0, z_1])$ if $h(z)$ is an extreme point of $S_1(z_2)$. Therefore to prove the theorem it suffices to show, when $h(z)$ is an extreme point of $S_1(z_2)$, that $h(z)$ is an extreme point of $S_1([z_0, z_1])$ if and only if $z_2 = z_0$ or $z_2 = z_1$. Let $h_m(z)$ be an extreme point of $S_1(z_2)$. Then

$$h_m(z) = \frac{D_m z - E_m z^m}{D_m - E_m z_2^{m-1}}, \ (m = 2, 3, \ldots).$$

Define

$$h_m(\eta, z) = \eta\left(\frac{D_m z - E_m z^m}{D_m - E_m z_0^{m-1}}\right) + (1 - \eta)\left(\frac{D_m z - E_m z^m}{D_m - E_m z_1^{m-1}}\right).$$

When $z_0 < z_2 < z_1$, we have $h_m(1, z) < h_m(z) < h_m(0, z)$ for z real and positive. Hence there exists a η_0, $0 < \eta_0 < 1$ such that $h_m(\eta_0, z) = h_m(z)$. This implies the coefficients of $h_m(\eta_0, z)$ agree with the coefficients of $h_m(z)$ for η_0 so that

$$\frac{\eta_0}{D_m - E_m z_0^{m-1}} + \frac{1 - \eta_0}{D_m - E_m z_1^{m-1}} = \frac{1}{D_m - E_m z_2^{m-1}}.$$

That is, $h_m(\eta_0, z) = h_m(z)$ throughout the unit disc E when

$$\eta_0 = \frac{D_m - E_m z_0^{m-1}}{D_m - E_m z_2^{m-1}}\left(\frac{z_1^{m-1} - z_2^{m-1}}{z_1^{m-1} - z_0^{m-1}}\right).$$

This shows that $h_m(z)$ is expressed as a linear combination of $f_m(z)$ and $g_m(z)$ when $z_0 < z_2 < z_1$. Hence $h_m(z)$ cannot be an extreme point of $S_1([z_0, z_1])$ when $z_0 < z_2 < z_1$ and $h_m(z) \in S_1(z_2)$.

Now we have only to show that $f_m(z)$ and $g_m(z)$ cannot be expressed, respectively, as a linear combination of extreme functions in $S_1(z)$, $z_0 < z \leqslant z_1$ and in $S_1(z)$, $z_0 \leqslant z < z_1$. This really follows from the following. For z positive and $0 \leqslant \eta \leqslant 1$, we have

$$f_m(z) < \eta\left(\frac{D_m z - E_m z^m}{D_m - E_m z_3^{m-1}}\right) + (1 - \eta)\left(\frac{D_m z - E_m z^m}{D_m - E_m z_4^{m-1}}\right)$$

$$(z_0 < z_3 \leqslant z_1, \ z_0 \leqslant z_4 \leqslant z_1)$$

and

$$g_m(z) > \eta \left(\frac{D_m z - E_m z^m}{D_m - E_m z_5^{m-1}} \right) + (1-\eta) \left(\frac{D_m z - E_m z^m}{D_m - E_m z_5^{m-1}} \right)$$

$$(z_0 \leqslant z_5 < z_1, \ \ z_0 \leqslant z_6 < z_1).$$

Hence the proof is completed.

Corollary 5.1—If $0 < z_0 < z_1 < 1$, the closed convex hull of $S_1(\{z_0, z_1\})$ is $S_1([z_0, z_1])$.

PROOF : Using the method of proof in Theorem 5.2, we can obtain the corollary.

In a similar way we can expand on the classes $K_1(z_0)$, $S_2(z_0)$ and $K_2(z_0)$ and prove the following.

Theorem 5.3—Let $\Gamma([z_0, z_1])$, $0 < z_0 < z_1 < 1$, denote any of the classes $S_2([z_0, z_1])$, $K_1([z_0, z_1])$ or $K_2([z_0, z_1])$. Then the extreme points of $T([z_0, z_1])$ are {extreme points of $T(z_0)$ $\cup$ extreme points of $T(z_1)$}, and the closed convex hull of $T(\{z_0, z_1\})$ is $T([z_0, z_1])$.

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L_p-APPROXIMATION BY SZÁSZ-MIRAKJAN-KANTOROVITCH OPERATORS

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We have proved that the sequence $\{\overset{*}{S}_n\}$ gives a linear approximation method on the normed space $(L_p\,[0,\infty),\ \|.\|_p)$ $(1 < p < \infty)$ and that the degree of approximation by this method is $\|f - \overset{*}{S}_n f\|_p = O\,(W_{1,p}\,(f\,;\,n^{-1/2}))$, where $W_{1,p}\,(f;\ .)$ is the first order of modulus of continuity with respect to the L_p norm.

1. INTRODUCTION

The Szász-Mirakjan operators[8] are defined as

$$S_n\,(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_{n,k}\,(x),\ p_{nk}\,(x) = \frac{e^{-nx}\,(nx)^k}{k!} \qquad \ldots(1.1)$$

and the Kantorovich variant[3] is

$$\overset{*}{S}_n\,(f;\,x) = \sum_{k=0}^{\infty} \left(n\textstyle\int_{I_k} f\,(t)\,dt\right)\ p_{n,k}\,(x),\ I_k = \left[\frac{k}{n},\ \frac{k+1}{n}\right]. \qquad \ldots(1.2)$$

For later reference we list useful relations :

$$n\ p_{n,k}\,(x)\int_{I_k} dt = p_{n,k}\,(x) \qquad \ldots(1.3)$$

and

$$n\int_0^{\infty} p_{n,k}\,(x)\,dx = 1.$$

2. L_p-APPROXIMATION

Given $f \in L_p\,[0,\infty)$, $1 \leqslant p < \infty$, we write $\overset{*}{S}_n\ f$ as a singular integral of the type

$$\overset{*}{S}_n\,(f;\,x) = \int_0^{\infty} H_n\,(x,\,t)\,f\,(t)\,dt$$

with the positive kernel

$$H_n(x, t) = n \sum_{k=0}^{\infty} p_{n,k}(x)\, l_{I_k}(t)$$

where l_{I_k} is the characteristic function of the interval I_k with respect $[0, \infty)$. Utilizing (1.3) and (1.4) we have for all n and x or t respectively

$$\int_{I_k} H_n(x, t)\, dt = \sum_{k=0}^{\infty} p_{n,k}(x) = 1 \qquad \text{...(2.1)}$$

$$\int_0^{\infty} H_n(x, t)\, dx = \sum_{k=0}^{\infty} l_{I_k}(t) = 1 \qquad \text{...(2.2)}$$

and thus by a theorem of Orlicz[6] follows easily that $\overset{*}{S}_n f$ belongs to $L_p[0, \infty)$ and the operator norms $\|\overset{*}{S}_n\|_p$ are uniformiy bounded by 1

Theorem 2.1—For $f \in L_p\ [0, \infty)$, $1 \leqslant p < \infty$, there holds

$$\lim_{n \to \infty} \| f - \overset{*}{S}_n f \|_p = 0. \qquad \text{...(2.3)}$$

PROOF : We show that (2.3) holds for the dense subspace $C[0, \infty)$ of $L_p[0, \infty)$. Using (1.3), we have for $f \in C[0, \infty)$ and an arbitrary $x \in [0, \infty)$

$$|\overset{*}{S}_n f(x) - S_n f(x)| \leqslant \sum_{k=0}^{\infty} n.\, p_{n,k}(x) \int_{I_k} \left| f(t) - f\left(\frac{k}{n}\right) \right| dt. \qquad \text{.. (2.4)}$$

In view of $\left| t - \frac{k}{n} \right| < \frac{1}{n}$ for $t \in I_k$ we obtain from (2.4) and (1.3)

$$|\overset{*}{S}_n f(x) - S_n f(x)| \leqslant W_{1,\infty}(f; 1/n)$$

and thus

$$\|\overset{*}{S}_n f(x) - S_n f(x)\| \leqslant W_{1,\infty}(f; 1/n). \qquad \text{...(2.5)}$$

Now

$$\| f - \overset{*}{S}_n f \|_p \leqslant \| f - S_n f \|_{\infty} + \| S_n f - \overset{*}{S}_n f \|_p.$$

For $n \to \infty$ each term goes to zero, which proves (2.3) for continuous functions. The rest of the proof follows by the density of $C[0, \infty)$ in $L_p[0, \infty)$.

3. DEGREE OF L_p-APPROXIMATION

The most efficient technique in deriving estimates for the degree of L_p-approximation in smoothing [1,4,5]. This means that f is first approximated by a function g with

g' in $L_p\,[0, \infty)$ and then g is approximaated by $\overset{*}{S}_n\, g$. The connection between these two processes is given via the K-functional of Peetre[7].

Theorem 3.1—For $g \in L_p^1\,[0, \infty)$, $p > 1$, there holds

$$\|g - \overset{*}{S}_n\, g\|_p \leqslant \frac{c_p}{\sqrt{n}}\, \|g'\| \qquad (n \geqslant 1) \qquad \text{...(3.1)}$$

where c_p is some positive constant, independent of g and n.

For its proof we need following :

Lemma 3.1—There exists a positive constant A, independent of $n \in N$ and $x \in [0, \infty)$, such that

$$\overset{*}{S}_n\,((t - x)^2; x) \leqslant A/n. \qquad \text{...(3.2)}$$

PROOF : Proof of the lemma is straight-forward.

Proof of Theorem 3.1—Fix $x \in [0, \infty)$. Then by (1.2) and (1.3)

$$| g\,(x) - \overset{*}{S}_n\, g(x) | = | \sum_{k=0}^{\infty} n\, p_{n,k}\,(x) \int_{I_k} \int_x^t g'\,(u)\, du\, dt\, |$$

$$\leqslant \theta_g'\,(x) \sum_{k=0}^{\infty} n\, p_{n,k}\,(x) \int_{I_k} | t - x |\, dt \qquad \text{...(3.3)}$$

where

$$\theta_g'\,(x) = \sup_{\substack{0 \leqslant t < \infty \\ t \neq x}} \frac{1}{t - x} \int_x^t | g'\,(u)\, du$$

is the Hardy-Littlewood majorant of g'. $g' \in L_p\,[0, \infty)$ implies for $p > 1$ by a theorem of Hardy and Littlewood (Zygmund[9], Theorems 13, 15) $\theta_{g'} \in L_p\,[0, \infty)$ with

$$\int_0^\infty \theta_{g'}^p\,(x)\, dx \leqslant 2 \left(\frac{p}{p - 1} \right)^p \int_0^\infty | g'\,(x) |^p\, dx. \qquad \text{...(3.4)}$$

Applying Cauchy-Schwarz's inequality and the lemma, we obtain from (3.3)

$$| g\,(x) - \overset{*}{S}_n\, g\,(x) | \leqslant \theta_g'\,(x) \{ \sum_{k=0}^{\infty} n\, p_{n,k}\,(x) \int_{I_k} (t - x)^2\, dx \}^{1/2}$$

$$\leqslant \sqrt{A}\, \theta_g'\,(x) \frac{1}{\sqrt{n}}$$

and this together with (3.4) for $p > 1$ gives

$$\|g - \overset{*}{S}_n\, g\|_p \leqslant \sqrt{A}\, \frac{p}{p-1}\, p \sqrt{2}\, \frac{1}{\sqrt{n}}\, \|g'\|_p$$

which completes the proof

In what follows we will measure smoothness by using the K-functional of Peetre[7]. It is for $f \in L_p[0, \infty)$, $1 \leqslant p < \infty$, defined by

$$K_p(t, f) = \inf_{g \in L_p^l} (\| f - g \|_p + t \| g' \|_p) \; (0 \leqslant t < \infty). \qquad \text{.. (3.5)}$$

The more classical measure for smoothness, the integral modulus of continuity, which for $f \in L_p[0, \infty)$, $1 \leqslant p < \infty$, is defined by

$$W_{1,p}(f, t) := \sup_{0 < h \leqslant t} \| f(. + h) - f(.) \|_p (I_h) \qquad \text{...(3.6)}$$

is in a certain sense equivalent to the K-functional. Johnen[2] (see his Prop. 6.1) proved that there are constants $c_1 > 0$ and $c_2 > 0$, independent of f and p, such that

$$c_1 W_{1,p}(f, t) \leqslant K_p(t, f) \leqslant c_2 W_{1,p}(f, t) \quad (0 \leqslant t < \infty). \qquad \text{...(3.7)}$$

Theorem 3.2—For $f \in L_p[0, \infty)$, $p > 1$, there holds

$$\| f - \overset{*}{S}_n f \| \leqslant M W_{1,p}\left(f, \frac{1}{\sqrt{n}}\right) \; (n \in N)$$

where M is some positive constant, independent of f and p.

Proof : In view of Theorem 3.1 and $\|\overset{*}{S}_n\| \leqslant 1$ $(n \in N, 1 \leqslant p < \infty)$ we have

$$\|h - \overset{*}{S}_n h\|_p \leqslant \begin{cases} 2 \| h \|_p, \; h \in L_p[0, \infty) \\ & (p > 1, n \geqslant 1). \\ \dfrac{c_p}{\sqrt{n}} \|h'\|_p \; h \in L_p^l [0, \infty) \end{cases}$$

When $f \in L_p(0, \infty)$, $p > 1$ and g is an arbitrary function from L_p^l $[0, \infty)$, $p > 1$ then

$$\|f - \overset{*}{S}_n f\|_p \leqslant \| (f - g) - \overset{*}{S}_n (f - g) \|_p + \|g - \overset{*}{S}_n g\|_p$$

$$\leqslant 2 (\| f - g\|_p + \frac{c_p}{\sqrt{n}} \|g'\|_p).$$

Taking now the infimum over all $g \in L_p^l$ $[0, \infty)$, $p > 1$ on the right hand side, using the definition of the K-functional and observing (3.6), we find

$$\| f - \overset{*}{S}_n f \|_p \leqslant 2 K_p\left(\frac{c_p}{\sqrt{n}}, f\right) \leqslant 2 c_2 W_{1,p}\left(f, \frac{c_p}{\sqrt{n}}\right)$$

$$\leqslant 2(1 + c_p) W_{1,p}\left(f, \frac{1}{\sqrt{n}}\right)$$

which complets the proof.

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EFFECT OF SUSPENDED PARTICLES ON THERMOSOLUTAL CONVECTION IN POROUS MEDIUM

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The thermosolutal convection in porous medium is considered to include the effect of suspended particles. For thermal Rayleigh number greater than or equal to solute Rayleigh number, the principle of exchange of stabilities is valid. The oscillatory modes may come into play if the thermal Rayleigh number is less than the solute Rayleigh number. The effect of suspended particles is to destabilize the layer. The medium permeability and the stable solute gradient have destabilizing and stabilizing effects respectively on the system. The rotation stabilizes a certain wavenumber range in thermosolutal convection in porous medium, which were unstable in the absence of rotation.

1. INTRODUCTION

A detailed account of the onset of Bénard convection, under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar[2]. Chandra[1] found that the instability depended on the depth of the layer. A Bénard-type cellular convection with fluid descending at the cell centre was observed when the predicted gradients were imposed, if the layer depth is more than 10 mm. The convection occurred at much lower gradients than predicted and appeared as irregular strips of elongated cells with fluid rising at the centre, if the depth of the layer was less than 7 mm. Chandra[1] observed this phenomenon in an air layer and called this motion 'columnar instability'. Thus there is a decades-old contradiction between the theory and the experiment. Scanlon and Segel[5] considered the effect of particle mass and heat capacity on the onset of Bénard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles. The effect of suspended particles was found to destabilize the layer. Palaniswamy and Purushotham[4] have considered the stability of shear flow of stratified fluids with fine dust and have found the effect of fine dust to increase the region of instability.

The problem of thermohaline convection in a layer of fluid heated and salted from below has been investigated by Veronis[6]. In such situation, buoyancy forces can arise not only from density differences due to variations in temperature, but also from those due to variations in solute concentration. The conditions under which convective

motions are important in geophysical situations are usually far removed from the consideration of single component fluid and therefore it is desirable to consider a fluid acted on by a solute gradient.

The medium has been considered to be non-porous in all the above studies. Lapwood[3] has studied the stability of convective flow in hydrodynamics in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by Wooding[7]. The gross effect, when the fluid slowly percolates through the pores of the rock, is represented by the well known Darcy's law.

The present paper deals with the effect of suspended particles on the thermosolutal convection in porous medium. The problem finds its usefulness in petroleum engineering, paper and pulp technology and several geophysical situations.

2. Formulation of the Problem and Perturbation Equations

Consider an infinite horizontal fluid-particle layer of thickness d bounded by the planes $z = 0$ and $z = d$ in porous medium. This layer is heated from below and subjected to a stable solute gradient such that a steady adverse temperature gradient β ($= |dT/dz|$) and a solute concentration gradient β' ($= |dC/dz|$) are maintained. Let ρ, μ, p, and $\vec{u}\,(u, v, w)$ denote respectively the density, the viscosity, the pressure and the velocity of the pure fluid; $\vec{v}\,(\bar{x}, t)$ and $N(\bar{x}, t)$ denote the velocity and number density of the particles, respectively. If k_1 is the medium permeability, g is the acceleration of gravity, $K = 6\pi\mu\eta$ where η is the particle radius, $\vec{v} = (1, r, s)$, $\bar{x} = (x, y, z)$ and $\vec{\lambda} = (0, 0, 1)$. Then the equations of motion and continuity for the fluid are

$$\frac{\rho}{\epsilon}\left[\frac{\partial \vec{u}}{\partial t} + \frac{1}{\epsilon}(\vec{u}.\nabla)\vec{u}\right] = -\nabla p - \rho g\vec{\lambda} - \frac{\mu}{k_1}\vec{u} + \frac{KN}{\epsilon}(\vec{v} - \vec{u}) \quad \text{..(1)}$$

$$\nabla.\vec{u} = 0. \quad \text{...(2)}$$

In the equations of motion (1) the presence of particles adds an extra force term, proportional to the velocity difference between particles and fluid. Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid; there must be an extra force term, equal in magnitude but opposite in sign, in the equation of motion for the particles. The buoyancy force on the particles is neglected. Interparticle reactions are also not considered for we assume that the distances between particles are quite large compared with their diameter. If mN is the mass of particles per unit volume, then the equations of motion and continuity for the particles, under the above assumptions, are

$$mN\left[\frac{\partial \vec{v}}{\partial t}+\frac{1}{\epsilon}(\vec{v}.\nabla)\vec{v}\right]=KN(\vec{u}-\vec{v})-\epsilon\, m\, Ng\,\vec{\lambda} \qquad \text{...(3)}$$

$$\epsilon\frac{\partial N}{\partial t}+\nabla.(N\vec{v})=0. \qquad \text{...(4)}$$

Let C, C_p, T, S, q and q' denote respectively the heat capacity of fluid at constant pressure, the heat capacity of particles, the temperature, the solute concentration, the "effective" thermal conductivity is the conductivity of the clean fluid and an analogous "effective" solute conductivity. If we assume that the particles and the fluid are in thermal and solute equilibrium, then the equations of heat and solute conduction give

$$[\rho\, C\, \epsilon+\rho_s\, C_s\,(1-\epsilon)]\frac{\partial T}{\partial t}+\rho\, C\,(\vec{u}.\nabla)\, T+mN\, C_p\left(\frac{\partial}{\partial t}+\vec{v}.\Delta\right)T$$

$$=q\,\nabla^2\, T \qquad \text{...(5)}$$

$$[\rho\, C\, \epsilon+\rho_s\, C_s\,(1-\epsilon)]\frac{\partial S}{\partial t}+\rho\, C\,(\vec{u}.\nabla)\, S+mNC_p\left(\frac{\partial}{\partial t}+\vec{v}.\nabla\right)S$$

$$=q'\,\nabla^2\, S \qquad \text{...(6)}$$

where ϵ is the medium porosity and ρ_s, C_s are the density and the heat capacity of the solid matrix respectively. The equation of state for the fluid is given by

$$\rho=\rho_0\,[1-a\,(T-T_0)+a'\,(S-S_0)]$$

where ρ_0 is the mean density of the clean fluid, a is the coefficient of thermal expansion and a' is the analogous coefficient of solvent expansion. The initial state of the system, denoted by subscript 0, is taken to be a quiescent layer with a uniform particle distribution N_0. Across the layer an adverse linear temperature gradient $-\beta$ and an adverse linear solute gradient $-\beta'$ are maintained. This initial state

$\vec{u}_0=0$, $\vec{v}_0=0$, $T_0=-\beta\, z$, $S_0=-\beta'\, z$, $N_0=$ constant is an exact solution to the governing equations. Let $\delta\rho$, δp, N, Θ, Γ, $\vec{u}$ and $\vec{v}$ denote the perturbations in density ρ_0, pressure p_0, number density N_0, temperature T, concentration S, fluid velocity (zero initially) and particle velocity (zero initially) respectively. The linearized dimensionless perturbation equations are

$$N_p^{-1}\frac{\partial \vec{u}}{\partial t}=-\nabla\delta p+N_R\,\Theta\,\vec{\lambda}-N_S\,\Gamma\,\vec{\lambda}-P^{-1}\vec{u}$$

$$+\,\omega\,(\vec{v}-\vec{u}) \qquad \text{...(7)}$$

$$\tau\frac{\partial \vec{v}}{\partial t}=\vec{u}-\vec{v} \qquad \text{...(8)}$$

$$\nabla . \vec{u} = 0, \frac{\partial M}{\partial t} + \nabla . \vec{v} = 0 \qquad ...(9)$$

$$E(1 + h) \frac{\partial \Theta}{\partial t} = (w + hs) + \nabla^2 \Theta \qquad ...(10)$$

$$E(1 + h) \frac{\partial \Gamma}{\partial t} = (w + hs) + \nabla^2 \Gamma . \qquad ...(11)$$

Physical variables have been scaled using d, $\frac{d^2}{k}$, $\frac{k}{d}$, $\frac{\rho \nu k}{d^2}$, βd and $\beta' d$ as the length, time, velocity, pressure, temperature and solute concentration scale factors respectively. ν is the kinematic viscosity of the fluid, k is the thermal diffusivity, $N_k = g\alpha\beta d^4/\nu k$ is the Rayleigh number, $N_S = \frac{g\alpha'\beta' d^4}{\nu k'}$ is the analogous solute Rayleigh number, $f = \frac{N_0 m}{\rho_0 \epsilon} = \tau\omega N_p$ is the mass fraction, $P = \frac{k_1}{d^2}$, $\omega = \frac{KN_0 d^2}{\rho_0 \nu \epsilon}$, $\tau = \frac{mk}{Kd^2}$ $h = \frac{f C_p}{C}$, $M = \frac{N}{N_0}$ and $N_p = \frac{\nu \epsilon}{k}$, and $E = \epsilon + \frac{(1 - \epsilon)}{\rho C} \rho_s C_s$. w and s are the vertical fluid and particle velocities respectively. In writing (7), use has been made of the Boussinesq equatión of state $\delta \rho = - \rho_0 (a \Theta - a'\Gamma)$.

Consider the case of two free surfaces having uniform temperature and solute concentration. Though the case of two free boundaries is little artificial. it enables us to find the analytical solution. The boundary conditions appropriate for the problem are

$$w = \frac{\partial^2 w}{\partial z^2} = \Theta = \Gamma = 0 \quad \text{at } z = 0 \text{ and } 1. \qquad ...(12)$$

Eliminating $\vec{v}$ and δp, the fluid, heat and solute equations become

$$\left(L_1 + \frac{L_2}{P} \right) \nabla^2 w = L_2 \left(N_R \nabla_1^2 \Theta - N_s \nabla_1^2 \Gamma \right) \qquad ...(13)$$

$$L_2 \left(E H \frac{\partial}{\partial t} - \nabla^2 \right) \Theta = \left(\tau \frac{\partial}{\partial t} + H \right) w \qquad ..(14)$$

$$L_2 \left(E H \frac{\partial}{\partial t} - \nabla^2 \right) \Gamma = \left(\tau \frac{\partial}{\partial t} + H \right) w \qquad ...(15)$$

where $L_1 = N_p^{-1} \left(\tau \frac{\partial^2}{\partial t^2} + F \frac{\partial}{\partial t} \right)$, $L_2 = \left(\tau \frac{\partial}{\partial t} + 1 \right)$, $F = f + 1$,

$$H = h + 1, \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ and } \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

Eliminating Θ, Γ between eqns. (13) — (15) we obtain

$$\left(L_1 + \frac{L_2}{P}\right)\left(EH\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = (N_R - N_S)\left(\tau\frac{\partial}{\partial t} + H\right)\nabla_1^2 w. \quad ...(16)$$

Analyze the perturbations in terms of normal modes by seeking solutions whose dependence on x, y and t is given by

$$W(z)\exp(i\alpha_x x + i\alpha_y y + nt) \quad ...(17)$$

where n is the growth rate and $\alpha = \left(\alpha_x^2 + \alpha_y^2\right)^{1/2}$ is the wave number of disturbance.

Using (17), eqn. (16) becomes

$$\left(L_1 + \frac{L_2}{P}\right)(D^2 - \alpha^2)(D^2 - \alpha^2 - EHn)W = (\tau n + H)(N_R - N_S)\alpha^2 W \quad ...(18)$$

where

$$L_1 = N_p^{-1}(\tau n^2 + Fn),\ L_2 = (1 + \tau n) \text{ and } D = \frac{d}{dz}.$$

3. Principle of Exchange of Stabilities and Oscillatory Modes

In this section we determine under what conditions the principle of exchange of stabilities is satisfied and the oscillations come into play.

Let

$$U = (D^2 - \alpha^2)W \quad ...(19)$$

and

$$X = \left(L_1 + \frac{L_2}{P}\right)U. \quad ..(20)$$

In terms of X, the equation satisfied by W is

$$(D^2 - \alpha^2 - Hn)X = \alpha^2(\tau n + H)(N_R - N_S)W. \quad ...(21)$$

Multiply eqn. (21) by X^*, the complex conjugate of X and integrate over the range of z. We obtain

$$\int_0^1 X^*(D^2 - \alpha^2 - Hn)X\,dz = \alpha^2(N_R - N_S)(\tau n + H)\int_0^1 X^* W dz \quad ....(22)$$

Integrating by parts and using eqns. (19), (20) and boundary conditions (12), eqn. (22) gives

$$I_1 + (\alpha^2 + EHn)\, I_2 = \alpha^2 (N_R - N_S)(\tau n + H)\left(L_1^* + \frac{L^*}{P} \right) I_3 \qquad (23)$$

where

$$I_1 = \int_0^1 | DX |^2 \, dz$$

$$I_2 = \int_0^1 | X |^2 \, dz \qquad \ldots(24)$$

$$I_3 = \int_0^1 (| DW |^2 + \alpha^2 | W |^2) \, dz.$$

The integrals I_1, I_2 and I_3 are positive definite. Putting $n = in_0$, where n_0 is real, into eqn. (23) and equating imaginary parts, we obtain

$$n_0^2 = - \frac{EHI_2 + \alpha^2 (N_R - N_S) \left(N_p^{-1} FH + \tau h p^{-1} \right) I_3}{\alpha^2 (N_R - N_S)\, N_p^{-1} \, \tau^2 \, I_3} \qquad \ldots(25)$$

or

$$n_0 = 0. \qquad \ldots(26)$$

Since the integrals are positive definite and n_0 is real, it follows that $n_0 = 0$ for all $N_R \geqslant N_S$. This establishes that n is real for $N_R \geqslant N_S$ and that the principle of exchange of stabilities is valid for the problem under consideration. For $N_R < N_S$, n_0^2 may be positive meaning thereby that the stable solute gradient may bring in oscillatory modes in the system.

4. Dispersion Relation and Discussion

We have proved that the marginal state is that of stationary convection and the principle of exchange of stabilities is valid for $N_R \geqslant N_S$. When instability sets in as stationary convection the marginal state will be characterized by $n = 0$ and eqn. (18) reduces to

$$1/P \, (D^2 - \alpha^2)^2 \, W = H (N_R - N_S) \, \alpha^2 \, W. \qquad \ldots(27)$$

Consider the case of two free boundaries. It can be shown that all the even order derivatives of W vanish on the boundaries and hence the proper solution of (27) characterizing the lowest mode is

$$W = W_0 \sin \pi z \qquad \ldots(28)$$

where W_0 is a constant.

Substituting the solution (28) in eqn. (27), we obtain

$$N_R = \frac{(\pi^2 + \alpha^2)^2}{HP\,\alpha^2} + N_S\,. \qquad \ldots(29)$$

It is evident from eqn. (29) that

$$\frac{dN_R}{dH} = -\frac{(\pi^2 + \alpha^2)^2}{H^2 P\alpha^2} \qquad \ldots(30)$$

and

$$\frac{dN_R}{dP} = -\frac{(\pi^2 + \alpha^2)^2}{HP^2\alpha^2}. \qquad \ldots(31)$$

The effect of suspended particles is thus to destabilize the layer. The medium permeability and the stable solute gradient have destabilizing and stabilizing effects respectively on the system.

5. Effect of Rotation

Here we consider the same problem as described above except that the system is in a state of uniform rotation $\vec{\Omega}\,(0, 0, \Omega)$. The Coriolis force on the particles is neglected. The linearized nondimensional perturbation equations of motion for the fluid are

$$N_p^{-1}\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\delta p - \frac{1}{P}u + \omega\,(l - u) + T_A^{1/2}\,v \qquad \ldots(32)$$

$$N_p^{-1}\frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}\delta p - \frac{1}{P}v + \omega\,(r - v) - T_A^{1/2}\,u \qquad \ldots(33)$$

$$N_p^{-1}\frac{\partial \omega}{\partial t} = -\frac{\partial}{\partial z}\delta p - \frac{1}{P}w + w\,(s - w) + N_R\,\Theta - N_S\,\Gamma \qquad \ldots(34)$$

where $T_A = 4\,\Omega^2 d^4/\,\nu^2\,\epsilon^2$ is the nondimensional number accounting for rotation and eqns. (8)—(11) remain unaltered.

Eliminating $\vec{v}$ and δp between eqns. (32) —(34), (8) and (9), we obtain

$$\left(L_1 + \frac{L_2}{P}\right)^2 \nabla^2 w + L_2^2\,T_A\,\frac{\partial^2 w}{\partial z^2} = L_2\left(L_1 + \frac{L_2}{P}\right)\nabla_1^2\,(N_R\,\Theta - N_S\,\Gamma). \qquad \ldots(35)$$

Eliminating Θ, Γ between eqns. (35), (10) and (11) and using expression (17), we obtain

$$(D^2 - \alpha^2 - EH\epsilon)\left[\left(L_1 + \frac{L_2}{P}\right)^2 (D^2 - \alpha^2) + L_2^2\,T_A\,D^2\right]W$$

(*equation continued on p.* 185)

$$= \left(L_1 + \frac{L_2}{P} \right) (N_R - N_S)\, \alpha^2 (\tau\epsilon + H)\, W. \qquad ...(36)$$

For the stationary convection, $n = 0$ and eqn. (36) reduces to

$$(D^2 - \alpha^2) \left[\frac{1}{P^2} (D^2 - \alpha^2) + T_A D^2 \right] W = \frac{1}{P} (N_R - N_S)\, \alpha^2 H W \qquad ...(37)$$

Here again we consider the case of two free boundaries with constant temperature and solute concentration. The appropriate solution for the problem is given by eqn. (28). Substituting the solution (28) in (37), we obtain

$$N_R = N_S + \frac{(\pi^2 + \alpha^2)^2}{P\alpha^2 H} + \frac{P T_A \pi^2 (\pi^2 + \alpha^2)}{\alpha^2 H}. \qquad ...(38)$$

It follows from eqn. (38) that

$$\frac{dN_R}{dN_S} = + 1 \qquad ...(39)$$

$$\frac{dN_R}{dh} = - \frac{(\pi^2 + \alpha^2) [1/P (\pi^2 + \alpha^2) + // pT_A \pi^2]}{\alpha^2 (1 + h)^2} \qquad ...(40)$$

and

$$\frac{dN_R}{dT_A} = + \frac{P\pi^2 (\pi^2 + \alpha^2)}{\alpha^2 (1 + h)} \qquad ...(41)$$

The stable solute gradient and rotation have stabilizing effects whereas the suspended particles have destabilizing effect on the system under consideration. Equation (38) also yields

$$\frac{dN_R}{dP} = \frac{(\pi^2 + \alpha^2)}{\alpha^2 H} \left[- \frac{(\pi^2 + \alpha^2)}{P^2} + T_A \pi^2 \right]. \qquad ...(42)$$

If $T_A > \left(1 + \frac{\alpha^2}{\pi^2}\right) \Big/ P^2$, $\frac{dN_R}{dP}$ is positive and if $T_A < \left(1 + \frac{\alpha^2}{\pi^2}\right) \Big/ P^2$, $\frac{dN_R}{dP}$ is negative. The medium permeability, therefore, has both stabilizing and destabilizing effects depending on the dimensionless rotation number. The rotation, thus, stabilizes a certain wave-number range in thermosolutal convection in porous medium in the presence of suspended particles, which were unstable in the absence of rotation.

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WAVE SOURCE POTENTIALS FOR TWO SUPERPOSED FLUIDS, EACH OF FINITE DEPTH

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Problems dealing with the generation of internal waves at the surface separating two fluids involves the consideration of different types of singularities in one of the two fluids. In this paper the velocity potentials describing line and point sources are obtained for the case when each fluid is of finite constant depth, neglecting effects of surface tension at the surface of separation.

1. INTRODUCTION

The study of surface waves in one fluid or internal waves in two fluids involves the consideration of singularities of different types in the fluids. In a previous paper[1] the velocity potentials describing these singularities were obtained, neglecting effects of surface tension at the surface of separation, for the cases when the lower fluid is of infinite and of finite constant depth while the upper being of infinite height. In another paper[2] the velocity potentials describing basic line and point multipoles were obtained when each fluid is of finite constant depth and the two superposed fluids be confined between rigid horizontal planes.

In this paper we discuss the basic line and point sources when each fluid is of finite constant depth. These time-harmonic singularities are described by harmonic potential functions which satisfy two linearized conditions at the surface of separation, and uniqueness is ensured by requiring that there are only outgoing waves in the far field. The method used is valid only for submerged singularities.

2. STATEMENT OF THE PROBLEM

We are concerned with the irrotational motion of two superposed non-viscous incompressible fluids under the action of gravity, neglecting any effect due to surface tension at the surface separating the two fluids. Each fluid is of infinite horizontal extent and if we take the origin O at the mean level of the interface and the axis Oy pointing vertically downwards into the lower fluid, let the two fluids be confined between rigid horizontal planes $y = h$, $y = -h'$.

The motion is simple harmonic with a small amplitude and angular frequency σ; it is due to an oscillating singularity in one of the two fluids. In two-dimensional

motion the singularity is a line wave source and in axisymmetric motion it is a point wave source. In each case, the velocity potentials of the lower and upper fluids are simple harmonic with period $2\pi/\sigma$ and it is more convenient to use the complex-valued potentials $\phi\, e^{-i\sigma t}$, $\phi'\, e^{-i\sigma t}$, of which the actual velocity potentials are the real parts. These potentials satisfy a boundary-value problem in which

$$\nabla^2 \phi = 0 \qquad \ldots(2.1)$$

$$\nabla^2 \phi' = 0 \qquad \ldots(2.2)$$

in the regions occupied by the fluids, except at the singularity ;

$$\frac{\partial \phi}{\partial y} = 0 \text{ on } y = h, \qquad \ldots(2.3)$$

$$\frac{\partial \phi'}{\partial y} = 0 \text{ on } y = -h' ; \qquad \ldots(2.4)$$

and the linearized interface conditions

$$\left.\begin{aligned} \frac{\partial \phi}{\partial y} &= \frac{\partial \phi'}{\partial y}, \\ K\phi + \frac{\partial \phi}{\partial y} &= s\left(K\phi' + \frac{\partial \phi'}{\partial y}\right) \end{aligned}\right\} \text{ on } y = 0 \qquad \ldots(2.5)$$

where $K = \sigma^2/g$ and s is the ratio of the density of the upper to that of the lower fluid.

These conditions are applied for each singularity considered. They are supplemented by the two general limiting conditions that ϕ or ϕ' behave like a typical singular harmonic function near the singularity, and the radiation condition that both functions represent outgoing waves in the far field.

3. Submerged Line Wave Source

Let the singularity be placed on the axis Oy at a distance f from 0. We define polar coordinates (R, θ) based on the singularity position by the equations

$$x = R \sin\theta, \; y \mp f = R \cos\theta,$$

according as the singularity is in the lower or upper fluid, so that R denotes the distance from the singularity.

(i) *Line Source Submerged in Lower Fluid*

Here ϕ and ϕ' are solutions of the boundary-value problem stated above with ϕ having a logarithmic singularity at $(0, f)$, i. e.

$$\phi \sim \log R \text{ as } R = \{x^2 + (y - f)^2\}^{1/2} \to 0.$$

Try as solutions the harmonic potentials

$$\phi = \log R - \frac{1 - s}{1 + s} \log R'$$

(equation continued on p. 188)

$$+ \int_0^\infty \{A(k) \cosh k(h - y) + B(k) \sinh ky\} \cos kx \, dk$$

$$\phi' = \frac{2}{1 + s} \log R + \int_0^\infty \{A'(k) \cosh k(h' + y) + B'(k) \sinh ky\} \cos kx \, dk$$

where $R' = \{x^2 + (y + f)^2\}^{1/2}$ is the distance from the image point $(0, - f)$. It is evident that ϕ, ϕ' as given above satisfy conditions (2.1) and (2.2).

Under suitable conditions the differentiation under the integral sign, and using the relations

$$\frac{\partial}{\partial y} (\log R) = \begin{cases} \int_0^\infty e^{-k(y-f)} \cos kx \, dk, & y > f, \\ - \int_0^\infty e^{-k(f-y)} \cos kx \, dk, & y < f, \end{cases}$$

$$\frac{\partial}{\partial y} (\log R') = \int_0^\infty e^{-k(y+f)} \cos kx \, dk, \quad y > - f,$$

conditions (2.3), (2.4) and (2.5) are satisfied if

$$B = - \frac{e^{-kh}}{k} \left(e^{kf} - \frac{1 - s}{1 + s} e^{-kf} \right) \operatorname{sech} kh$$

$$B' = \frac{2}{(1 + s)k} e^{-k(f+h')} \operatorname{sech} kh'$$

$$k(A \sinh kh + A' \sinh kh') = - e^{-k(h-f)} \operatorname{sech} kh$$

$$+ \frac{e^{-kf}}{1 + s} \{ (1 - s) e^{-kh} \operatorname{sech} kh - 2 e^{-kh'} \operatorname{sech} kh' \}$$

$$K A \cosh kh - \{s K \cosh kh' - (1 - s) k \sinh kh'\} A'$$

$$= \frac{2(1 - s)}{1 + s} e^{-kf} \tanh kh'.$$

These determine, A, B A' and B' which when substituted in the above assumed forms, give

$$\phi = \log R - \frac{1 - s}{1 + s} \log R'$$

$$+ \frac{1}{1+s} \int_0^\infty \frac{1}{k\Delta} \Big[\left(k \sinh kh' - \frac{sM}{1 + s} e^{-kh'}\right) e^{k(h-f-y)}$$

$$- e^{-k(h-f)} \{(s M \cosh kh' - (1 + s) k \sinh kh') \cosh ky$$

$$+ M \sinh kh' \sinh ky\} + e^{-k(h+f)} \{ k \sinh kh'$$

(equation continued on p. 189)

$$+ \frac{M}{1+s} (\sinh kh' - s \cosh kh') \Big\} (\sinh ky + s \cosh ky) \Big]$$

$$\cos kx \, dk \qquad \ldots(3.1)$$

$$\phi' = \frac{2}{1+s} \log R + \frac{1}{1+s} \oint_0^\infty \frac{1}{k\Delta} [\{ (k + K) e^{-k(h+f)} - k e^{k(h-f)}$$

$$- M e^{-k(h-f)}\} \cosh k (h' + y) + 2e^{-k(h'+f)} \{k \sinh kh \cosh ky$$

$$- \frac{M}{1+s} (\cosh kh \cosh ky - s \sinh kh \sinh ky) \Big\}\Big] \cos kx \, dk$$

$$\ldots(3.2)$$

where

$$\Delta (k) = \frac{M}{1+s} (\cosh kh \sinh kh' + s \sinh kh \cosh kh')$$

$$- k \sinh kh' \sinh kh'$$

$$= \tfrac{1}{4} (k + K) e^{-k(h-h')} + \tfrac{1}{4} (k - K) e^{k(h-h')}$$

$$- \tfrac{1}{4} (k + M) e^{-k(h+h')} - \tfrac{1}{4} (k - M) e^{k(h+h')} \qquad \ldots(3.3)$$

$$M = (1 + s) K / (1 - s).$$

Now, $\Delta (k)$ has one simple zero at $k = m$, say, on the real axis of k. This introduces simple poles for the integrals in ϕ, ϕ'. Below this pole we make an indentation of the contours of integrations in (3.1) and (3.2). By putting $2 \cos kx = e^{ik|x|} + e^{-ik|x|}$, and rotating the contours in the indented integrals in ϕ, ϕ' into contours in the first and fourth quadrants so that we must include the residue term at $k = m$ for the first, these integrals tend to

$$- C_1 \frac{\cosh m (h - y)}{m \sinh mh} e^{im|x|}, \; C_1 \frac{\cosh m (h' + y)}{m \sinh mh'} e^{im|x|},$$

as $|x| \to \infty$, where

$$C_1 = \frac{4\pi i}{1+s} \Big[\{(m + k) e^{-m(h+f)} - m e^{m(h-f)} - M e^{-m(h-f)}\} \sinh mh'$$

$$+ \frac{2sM}{1+s} \sinh mh \, e^{-m(h'+f)} \Big] \Big/ [e^{-m(h-h')} + (2hm - 2hm - 1)$$

$$\times e^{m(h+h')} + (2h' M + 2h' m - 1) e^{-m(h+h')} + (2 (h - h') (m - K)$$

$$+ 1) e^{m(h-h')}] = H \Big[\{(m + k) e^{-m(h+f)} - m e^{m(h-f)} - M e^{-m(h-f)}\}$$

$$\times \sinh mh' + \frac{2s M}{1+s} \sinh mh \, e^{-m(h'+f)} \Big] \qquad \ldots(3.4)$$

where

$$H = \frac{4\pi i}{1+s}\,[e^{-m(h-h')} + (2hM - 2hm - 1)\,e^{m(h+h')}$$

$$+ (2h' M + 2h' m - 1)\,e^{-m(h+h')} + (2\,(h - h')\,(m - K) + 1)$$

$$\times\, e^{m(h-h')})]^{-1}. \qquad \ldots(3.5)$$

(ii) *Line Source Submerged in Upper Fluid*

The boundary-value problem for this case is similar to the previous one except that, now the singularity is at $(0, -f)$ in the upper fluid, and we have

$$\phi' \sim \log R \text{ as } R = \{x^2 + (y+f)^2\}^{1/2} \to 0.$$

We try as solutions

$$\phi = \frac{2s}{1+s}\log R + \int_0^\infty \{A\,(k)\cosh k\,(h-y) + B\,(k)\sinh ky\}\cos kx\,dk$$

$$\phi' = \log R + \frac{1-s}{1+s}\log R'$$

$$+ \int_0^\infty \{A'\,(k)\cosh k\,(h'+y) + B'\,(k)\sinh ky\}\cos kx\,dk$$

where $R' = \{x^2 + (y-f)^2\}^{1/2}$ is the distance from the image point $(0, f)$, and use the relations

$$\frac{\partial}{\partial y}(\log R) = \begin{cases} \int_0^\infty e^{-k(y+f)}\cos kx\,dk, & y > -f, \\ -\int_0^\infty e^{k(y+f)}\cos kx\,dk, & y < -f, \end{cases}$$

$$\frac{\partial}{\partial y}(\log R') = -\int_0^\infty e^{-k(f-y)}\cos kx\,dk,\; y < f.$$

Conditions (2.3), (2.4) and (2.5) are satisfied if

$$B = -\frac{2s}{(1+s)k}\,e^{-k(h+f)}\operatorname{sech} kh$$

$$B' = \frac{e^{-kh'}}{k}\left(e^{kf} + \frac{1-s}{1+s}\,e^{-kf}\right)\operatorname{sech} kh'$$

$$k\,(A\sinh kh + A'\sinh kh') = -\,e^{-k(h'-f)}\operatorname{sech} kh'$$

$$-\frac{e^{-kf}}{1+s}\,\{(1-s)\,e^{-kh'}\operatorname{sech} kh'$$

$$+ 2s\,e^{-kh}\operatorname{sech} kh\},$$

$$\{K \cosh kh - (1 - s) k \sinh kh\} A - s K A' \cosh kh'$$
$$= -\frac{2s(1-s)}{1+s} e^{-kf} \tanh kh.$$

These lead to

$$\phi = \frac{2s}{1+s} \log R + \frac{s}{1+s} \mathscr{P}\int_0^\infty \frac{1}{k\Delta} \Big[\{ (k - K) e^{-k(h'+f)} - k e^{k(h'-f)}$$
$$- M e^{-k(h'-f)}\} \cosh k(h-y) + 2e^{-k(h+f)} \{k \sinh kh' \cosh ky$$
$$- \frac{M}{1+s} (\sinh kh' \sinh ky + s \cosh kh' \cos ky) \}\Big] \cos kx \, dk \qquad \text{...(3.6)}$$

$$\phi' = \log R + \frac{1-s}{1+s} \log R' + \frac{1}{1+s} \mathscr{P}\int_0^\infty \frac{1}{k\Delta} [s (k \sinh kh$$
$$- \frac{M}{1+s} e^{-kh}) e^{k(h'-f+y)} - e^{-k(h'-f)} \{(M \cosh kh - (1+s) k$$
$$\sinh kh) \cosh ky - s M \sinh kh \sinh ky\}$$
$$+ e^{-k(h+f)} \left\{ k \sinh kh - \frac{M}{1+s} (\cosh kh - s \sinh kh)\right\}$$
$$(\cosh ky - s \sinh ky)] \cos kx \, dk \qquad \text{...(3.7)}$$

where Δ is given by (3.3).

These potentials have the outgoing waves

$$C_1' \frac{\cosh m(h-y)}{m \sinh mh} e^{\mathrm{i}m|x|}, \; - C_1' \frac{\cosh m(h'+y)}{m \sinh mh'} e^{\mathrm{i}m|x|}$$

as $|x| \to \infty$, where

$$C_1' = s H\left[\left\{(m - K) e^{-m(h'+f)} - m e^{m(h'-f)} - M e^{-m(h'-f)}\right\} \sinh mh\right.$$
$$\left. + \frac{2M}{1+s} \sinh mh' e^{-m(h+f)}\right] \qquad \text{.. (3.8)}$$

obtained as in the previous case and H is given by (3.5).

4. Submerged Point Wave Source

We now define cylindrical polar coordinates (r, ψ, y), where r is the distance from the y — axis, and also spherical polar coordinates (R, θ, ψ) based on the singularity position, by the equations

$$r = R \sin \theta, \quad y \mp f = R \cos \theta.$$

R therefore denotes the distance from the singularity. We consider only point singularities for which Oy is an axis of symmetry, so that the velocity potentials ϕ, ϕ' are independent of the angle ψ.

The boundary value problem for ϕ, ϕ' in this case is given by eqns. (2.1) — (2 5), supplemented by the two limiting conditions near the singularity and in the far field.

(i) *Point source Submerged in lower fluid*

Here

$$\phi \sim 1/R \text{ as } R = \{r^2 + (y-f)^2\}^{1/2} \to 0.$$

If we try as solutions

$$Q = \frac{1}{R} + \int_0^\infty \{A(k) \cosh k(h-y) + B(k) \sinh ky\} J_0(kr)\, dk,$$

$$\phi' = \int_0^\infty A'(k) \cosh k(h'+y) J_0(kr)\, dk$$

then using the representations

$$\frac{1}{R} = \begin{cases} \int_0^\infty e^{-k(y-f)} J_0(kr)\, dk, & y > f, \\ \int_0^\infty e^{k(y-f)} J_0(kr)\, dk, & y < f, \end{cases}$$

conditions (2.3), (2.5) determine A, A', B which when substituted in the above assumed forms, give

$$\phi = \frac{1}{R} - \oint_0^\infty \frac{1}{\Delta} \Big[\Big\{ k \sinh kh' + \frac{M}{1+s} (\sinh kh' - s \cosh kh')\Big\}$$
$$e^{-kf} \cosh k(h-y) + e^{-k(h-f)} \Big\{ k \sinh kh' \cosh ky$$
$$- \frac{M}{1+s} (\sinh kh' \sinh ky + s \cosh kh' \cosh ky)\Big\}\Big] J_0(kr)\, dk, \qquad \ldots(4.1)$$

$$\phi = \frac{2M}{1+s} \oint_0^\infty \frac{1}{\Delta} \cosh k(h-f) \cosh k(h'+y) J_0(kr)\, dk \qquad \ldots(4.2)$$

where Δ is given by (3.3). Notice that the velocity potentials ϕ, ϕ' given by (4.1), (4.2) could be deduced directly by putting $n = 0$ in the results of point multipoles submerged in lower fluid obtained in (2). The path of integration is indented below the simple pole at $k = m$ to give the diverging cylindrical waves

$$\phi \sim - C_2 \frac{\cosh m(h-y)}{\sinh mh} H_0^{(1)}(mr),$$

$$\phi' \sim C_2 \frac{\cosh m(h'+y)}{\sinh mh'} H_0^{(1)}(mr),$$

as $r \to \infty$, where

$$C_2 = 2\, M\, H \cosh m\,(h - f) \sinh mh', \qquad \ldots(4.3)$$

and H is given by (3.5).

(ii) *Point Source Submerged in Upper Fluid*

In this case

$$\phi' \sim 1/R \text{ as } R = \{r^2 + (y + f)^2\}^{1/2} \to 0.$$

Try as solutions the harmonic potentials

$$\phi = \int_0^\infty A\,(k) \cosh k\,(h - y)\, J_0\,(kr)\, dk,$$

$$\phi' = \frac{1}{R} + \int_0^\infty \{A'\,(k) \cosh k\,(h' + y) + B'\,(k) \sinh ky\}\, J_0\,(kr)\, dk.$$

As before, these lead to

$$\phi = \frac{2sM}{1 + s} \oint_0^\infty \frac{1}{\Delta} \cosh k\,(h' - f) \cosh k\,(h - y)\, J_0\,(kr)\, dk \qquad \ldots(4.4)$$

$$\phi' = \frac{1}{R} - \oint_0^\infty \frac{1}{\Delta}\Bigg[\Big\{k \sinh kh - \frac{M}{1 + s} (\cosh kh - s \sinh kh)\Big\}$$

$$e^{-kf} \cosh k\,(h' + y) + e^{-k(h-f)} \Big\{ k \sinh kh \cosh ky$$

$$- \frac{M}{1 + s} (\cosh kh \cosh ky - s \sinh kh \sinh ky) \Big\}\Bigg] J_0\,(kr)\, dk$$

$$\ldots(4.5)$$

where Δ is given by (3.3) and in this case we obtain the diverging cylindrical waves

$$\phi \sim C_2' \frac{\cosh m\,(h - y)}{\sinh mh} H_0^{(1)}\,(mr),$$

$$\phi' \sim - C_2' \frac{\cosh m\,(h' + y)}{\sinh mh'} H_0^{(1)}\,(mr),$$

as $r \to \infty$, where

$$C_2' = 2s\, M\, H \cosh m\,(h' - f) \sinh mh \qquad \ldots(4.6)$$

and H is given by (3.5).

5. Conclusion

Known results in the absence of the upper fluid can be made evident by putting $s = 0$ in the suitable formulae for ϕ. Also the specia lcases $h' \to \infty$ or $h, h' \to \infty$ give previously known results.

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ON THE PRECESSION OF THE PERIHELION OF MERCURY

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The possibility of obtaining the general relativistic effects without using the principles of General Relativity is an intriguing one. Recently, Bagge claimed to have obtained the correct value of the precession of the perihelion of Mercury within the framework of Special Relativistic dynamics alone. A critical examination of this approach reveals certain drawbacks which are reported in this paper.

1. Introduction

In an article, Roxburgh[1] demonstrated that the concept of space-time curvature in General Relativity has the status of a mere convention. According to ref. 1, one can look upon a flat space-time with a field embedded in it as a curved space-time with no field. This idea justifies the recent efforts to obtain the so-called General Relativistic effects without the use of any space-time metric whatsoever. Some such efforts have proved successful[2,3] but not all. For instance, Bagge[4] suggested recently that one can obtain, solely within the framework of Special Relativistic dynamics, the correct precession of the Perihelion of Mercury. The undertaking seems to have been prompted by the prospect of a possible analogy of this problem with the precessing motion of electron around the nucleus of Hydrogen atom. As is well-known, Sommerfeld[5], while calculating the precession of electron, started with the following eqation

$$\frac{d}{dt}\,\frac{m_0\bar{v}}{(1-\beta^2)^{1/2}} = \frac{Ze^2}{r^3}\,\bar{r},\ \beta = \frac{v}{c} \qquad \ldots(1)$$

where the terms have their usual meanings.

Analogously if one merely replaces the r. h. s. by $GMm_0\, r^{-3}\,\bar{r}$ (M = solar mass, m_0 = mercury mass, G = gravitational constant), one obtains, as usual, a precession of

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the Perihelion of Mercury, the numerical value being only 1/6th of the observed value. Bagge pointed out that the relativistic variation of mass of Mercury should also be incorporated in the force term on the r. h. s. of Sommerfeld's equation (1). He chooses the transverse mass transformation and thus obtains a precession of 42.087″/century. The present paper aims to show that his approach, though interesting, does "not" yield the claimed value.

2. Precession

In Bagge[4] the problem was tackled first analytically and then by numerical computations. In the analytic approach, the author introduced the relativistic mass in the energy equation and started with the equation

$$E = \frac{m_0 c^2}{(1 - \beta^2)^{1/2}} - m_0 c^2 - \frac{GMm_0}{r(1 - \beta^2)^{1/2}} \qquad ...(2)$$

where the last term corresponds to potential energy. In the numerical approach, he starts with the equation

$$\frac{d}{dt} \frac{m_0 \bar{v}}{(1 - \beta^2)^{1/2}} = \frac{GMm_0}{(1 - \beta^2)^{1/2}} \cdot \frac{\bar{r}}{r^3} \qquad ...(3)$$

where v is the velocity of Mercury. But evidently, the two equations of motion (2) and (3) are not equivalent since (2) is "not" the integral of (3). This is not surprising since velocity dependent forces, appearing in the r. h. s. of (3) do not correspond to to a potential energy function in general. Naturally, there arises a confusion with regard to the choice of the starting equation. Starting with (2) in his analytic approach, Bagge obtained a precession of 21″/century to a first approximation. He believed that the correct value would follow if higher order terms are taken into account and thus considered (3), as if for convenience, for numerical computations. In this, as pointed out above, he actually considered a different equation of motion. However, solving eqn. (3) analytically, and also numerically, we find that it yields a precession of only 14″/century and not 42.087″/century.

Writing out (3) in polar coordinates (r, φ) we get

$$\left\{\frac{d}{dt}\left[\frac{m_0 \dot{r}}{(1 - \beta^2)^{1/2}}\right] - \frac{r\dot{\varphi}^2}{(1 - \beta^2)^{1/2}}\right\}\hat{e}_r + \left\{\frac{d}{dt}\frac{r\dot{\varphi}}{(1 - \beta^2)^{1/2}} + \frac{\dot{r}\dot{\varphi}}{(1 - \beta^2)^{1/2}}\right\}\hat{e}_\varphi$$

$$= \frac{GM}{r^2(1 - \beta^2)^{1/2}}\hat{e}_r. \qquad ...(4)$$

Equating the coefficients of e_φ to zero, we get the constant of motion as

$$\frac{r^2 \dot{\varphi}}{(1 - \beta^2)^{1/2}} = S = \frac{p}{m_0} \qquad ...(5)$$

where p is the angular momentum of Mercury. Eliminating the time derivative through (5) and putting $u = 1/r$, we get from (4),

$$\frac{d^2 u}{d\varphi^2} + u = \frac{GM}{S^2(1-\beta^2)}. \qquad \text{...(6)}$$

Also, using (5), we get

$$(1-\beta^2)^{-1} = \left(1 - \frac{\dot{r}^2 + r^2\dot{\varphi}^2}{c^2}\right)^{-1} = 1 + \frac{S^2}{c^2}\left[\left(\frac{du}{d\varphi}\right)^2 + u^2\right] \qquad \text{...(7)}$$

so that (6) becomes

$$\frac{d^2u}{d\varphi^2} + u = b\left(1 + \mu\left[u^2 + \left(\frac{du}{d\varphi}\right)^2\right]\right) \qquad \text{...(8)}$$

where

$$b = \frac{GM}{S^2}, \; \mu = \frac{GM}{c^2}\, b \qquad \text{...(9)}$$

and the last term inside the square bracket represents the perturbation in the Keplerian orbit of Mercury due to relativistic considerations. The Perihelion precession can now be obtained in the usual[6] way. Set

$$u = b + \mu\beta_0 + b\,\epsilon\cos\rho\varphi + \mu\sum_{2}^{\infty}\beta_\nu\cos\nu\rho\varphi \qquad \text{...(10)}$$

where β's are constants, ϵ is the eccentricity and ρ determines the precession. Substituting (10) into (8) and comparing coefficients of $\cos\rho\varphi$, we obtain

$$\rho = (1 - 2\mu b^2)^{1/2}. \qquad \text{...(11)}$$

The precession per revolution is given by

$$\Delta\varphi = 2\pi(\rho^{-1} - 1) \simeq \frac{2\pi\, G^2 M^2 m_0^2}{p^2 c^2} \qquad \text{...(12)}$$

which is equivalent to only 14″/century. However, if the force field on the r. h. s. of (3) is replaced by

$$\frac{GMm_0}{(1-\beta^2)^{5/2}} \cdot \frac{\bar{r}}{r^3}$$

one analogously obtains the correct value of the precession 42″/century. But this replacement does not seem to have an immediate physical significance.

3. Numerical Computations

In spite of the above, numerical calculations yield in Bagge[4] value of 42.087″/century which is surprising. It is therefore necessary to look into the numerical computations. Instead of starting with Bagge's dynamical equations (time dependent), it is obviously

advantageous to consider the path eqn. (8) itself. In this way, we can directly obtain the locus (r vs. φ) without going through an intermediary time parameter t and thereby eliminating one possible source of error.

To begin with, let us note that the actual locus is essentially a small pertubation to the Keplerian orbit given by the differenrial equation

$$\frac{d^2u_k}{d\varphi^2} + u_k = b \quad \text{.. (13)}$$

the solution being

$$u_k = \frac{1 + e\cos\varphi}{a(1 - e^2)} \quad \text{...(14)}$$

a = semi-major axis, e = eccentricity. We choose to start with the perturbation equation

$$\frac{d^2\epsilon}{d\varphi^2} + \epsilon = b\mu\left[(\epsilon + u_k)^2 + \left(\frac{d\epsilon}{d\varphi} + \frac{du_k}{d\varphi}\right)^2\right] \quad \text{...(15)}$$

in which $\epsilon = u - u_k$ is the perturbation function. Since u_k is known in the closed form (14), we expect a better accuracy in our results.

Normalizing u_k and ϵ, one obtains

$$\frac{d^2E}{d\varphi^2} + E = \frac{b\mu}{d}\left[(E + U_k)^2 + \left(\frac{dE}{d\varphi} + \frac{dU_k}{d\varphi}\right)^2\right] \quad \text{...(16)}$$

where $E = a\epsilon$ and $U_k = au_k$. In terms of the known period T of Mercury's orbit

$$\frac{b\mu}{a} = \left(\frac{2\pi a}{Tc}\right)^2.$$

The numerical values of the relevant constants are :

$$\begin{aligned} e &= 0.206 \\ a &= 5.791 \times 10^{12}\ \text{cm} \\ \tau &= 7.600487 \times 10^{6}\ \text{sec} \\ c &= 3 \times 10^{10}\ \text{cm/sec} \end{aligned} \quad \text{...(17)}$$

and the boundary conditions are :

$$\text{at } \varphi = 0;\ E = 0 \text{ and } \frac{dE}{d\varphi} = 0. \quad \text{...(18)}$$

The problem is now ready for programming. A double precision programming (Runge-Kutta fourth order) was fed to NELCO-4200 computer. The range of φ between 0° to 360° was divided into n parts where n = 100, 500, 5000, 9999 and beyond φ = 360°

TABLE I

n	$\varphi - 2\pi$ in sec.	$dU/d\varphi$
	0.032	0.2547294640588721D — 08
100	0.034	0.4613431431388803D — 09
	0.036	−0.1624608354310967D — 08
	0.032	0.2570015790537455D — 08
500	0.034	0.4840642843311861D — 09
	0.036	−0.1601887221875089D — 08
	0.032	0.2570896821031468D — 08
5000	0.034	0.4849453128542592D — 09
	0.036	−0.1601006195322956D — 08
	0.032	0.2571094797305642D — 08
9999	0.034	0.4851432890189409D — 09
	0.036	−0.1600808219267760D — 08

(where the minimum is expected to occur), the increment in φ was taken to be 0.002″. The printout consists of φ vs. $\frac{dV}{d\varphi}$ (where $U = au$) and in Table I above we show only $\frac{dU}{d\varphi}$ vs. φ around the φ values where a change of sign in $\frac{dU}{d\varphi}$ occurs.

It is evident from the table that $\frac{dU}{d\varphi} = 0$ between $\varphi = 360° - 0' - 0.034''$ to $\varphi = 360° - 0' - 0.036''$. This conclusion does not dep endon whether $n = 100$ or 100 times of that. Therefore, the total precession $\Delta\varphi'$ per century must lie between 14.11″ to 14.94″.

Now we show below that Bagge's computer calculations contradict his own claim and support only the above conclusion. Equations (35) and (36) of the first paper of Bagge[4] give

$$\Delta\varphi = \Delta\varphi^* I_{min} = 0.0000282°. \qquad \text{...(19)}$$

This is supposed to represent the position of the minimum after $\varphi = 359.9999814°$ corresponding to the division $n = 12960000$ (from Bagge's data). Thus, the minimum occurs after one revolution at

$$\begin{aligned}\varphi + \Delta\varphi &= 359.9999814° + \Delta\varphi \\ &= 360.0000096°.\end{aligned} \qquad \text{...(20)}$$

Hence the shift of Perihelion ($\Delta\varphi'$) after one revolution is

$$\Delta\varphi' = \varphi + \Delta\varphi - 360^\circ = 0.03456''. \qquad \ldots(21)$$

This corresponds only to 14.34″ of precession per century! However, the method in first paper of Bagge[4] has a merit in that it admits of revisions, for instance, in the ponderomotive force term as already indicated in section 2. Certainly, this method has opened a fresh line of thought in the direction of search for an alternative to Einstein's theory of General Relativity.

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